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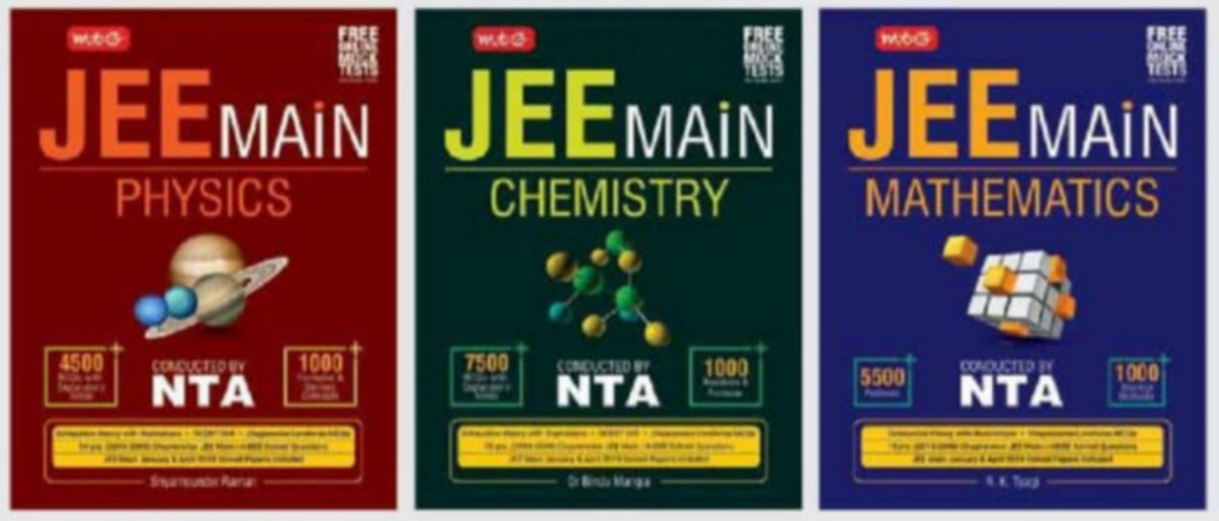

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MATHEMATICS today

Vol. XXXVIII No. 8 August 2020

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e-mail : info@mtg.in website : www.mtg.in

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Printed and Published by Mahabir Singh on behalf of MTG Learning Media Pvt. Ltd. Printed at HT Media Ltd., B-2, Sector-63, Noida, UP-201307 and published at 406, Taj Apartment, Ring Road, Near Safdarjung Hospital, New Delhi - 110029.
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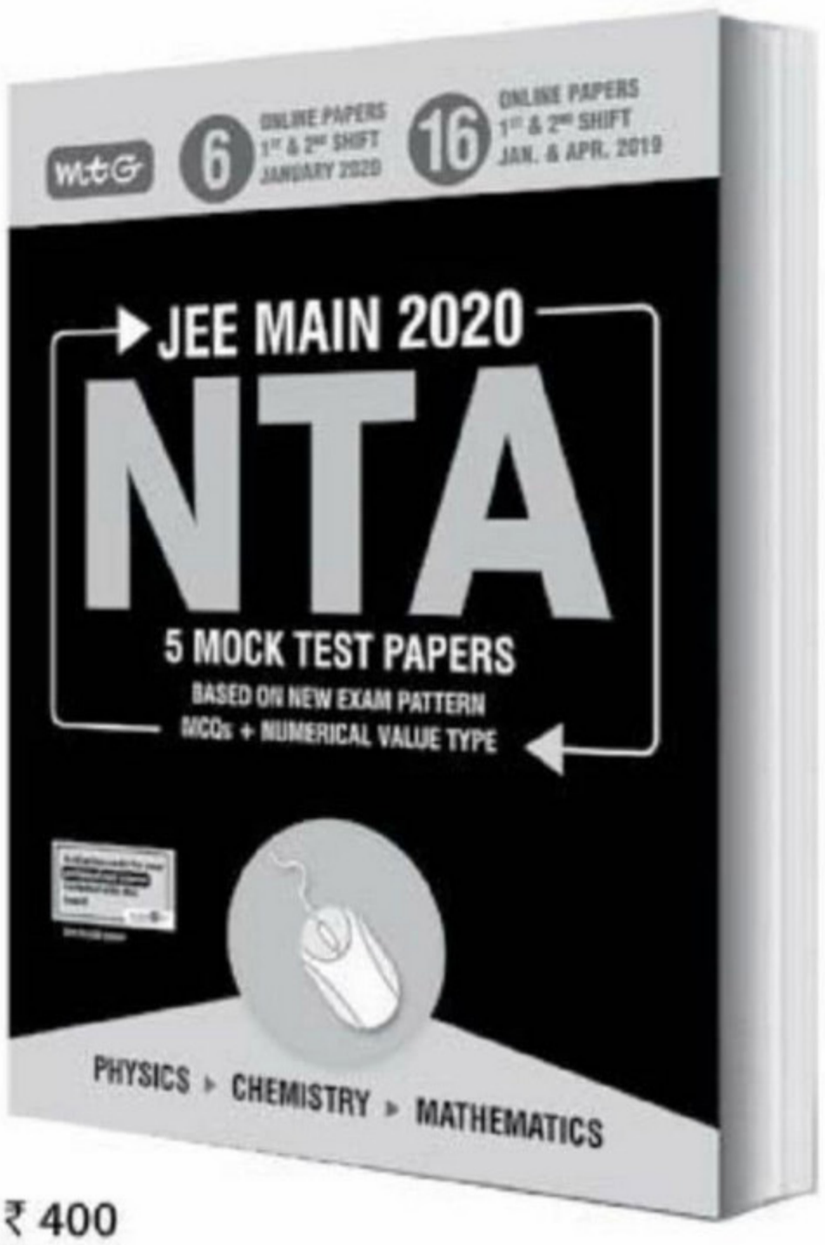

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GEAR UP FOR JEE MAIN

EXAM ON 1st - 6th September 2020

with Numerical Value Type Questions

1. If tan$A$, tan$B$ are the roots of $x^2 - Px + Q = 0$ (where $P, Q \in R$), then the value of $\sin^2(A+B) =$

(a) $\frac{P^2}{P^2+(1-Q)^2}$ (b) $\frac{P^2}{P^2+Q^2}$

(c) $\frac{Q^2}{P^2+(1-Q)^2}$ (d) $\frac{P^2}{(P+Q)^2}$

2. If $a_k = \frac{1}{k(k+1)}$, for $k = 1,2,3,.....,n$, then $\left(\sum_{k=1}^{n} a_k\right)^2 =$

(a) $\frac{n}{n+1}$ (b) $\frac{n^2}{(n+1)^2}$

(c) $\frac{n^4}{(n+1)^4}$ (d) $\frac{n^6}{(n+1)^6}$

3. $\sum_{k=1}^{\infty} \frac{1}{k!}\left(\sum_{n=1}^{k} 2^{n-1}\right) =$

(a) e (b) $e^2 + e$ (c) e^2 (d) $e^2 - e$

4. The sum of the series $\frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + ...$ to infinite terms, if $|x| < 1$ is

(a) $\frac{x}{1-x}$ (b) $\frac{1}{1-x}$ (c) $\frac{1+x}{1-x}$ (d) 1

5. If $a = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$, then $\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{vmatrix}$ is

(a) purely real (b) purely imaginary

(c) $a\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ (d) a

6. $\begin{vmatrix} {}^{x}C_r & {}^{x}C_{r+1} & {}^{x}C_{r+2} \\ {}^{y}C_r & {}^{y}C_{r+1} & {}^{y}C_{r+2} \\ {}^{z}C_r & {}^{z}C_{r+1} & {}^{z}C_{r+2} \end{vmatrix} - \begin{vmatrix} {}^{x}C_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^{y}C_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^{z}C_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix} =$

(a) 0 (b) 2^n

(c) ${}^{x+y+z}C_r$ (d) ${}^{x+y+z}C_{r+2}$

7. If A and B are square matrices of the same order and A is non-singular, then for a positive integer n, $(A^{-1}BA)^n$ is equal to

(a) $A^{-n}B^nA^n$ (b) $A^nB^nA^{-n}$

(c) $A^{-1}B^nA$ (d) $n(A^{-1}BA)$

8. If A and B are any two different square matrices of order n with $A - B$ is non-singular and $A^3 = B^3$ and $A(AB) = B(BA)$, then

(a) $A^2 + B^2 = O$ (b) $A^2 + B^2 = I$

(c) $A^2 + B^3 = I$ (d) $A^3 + B^3 = O$

9. If $x = \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}$ and

$y = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}$, then $x^2 + y^2 =$

(a) 1 (b) 2 (c) 3 (d) 4

10. The number of values of y in $[-2\pi, 2\pi]$ satisfying the equation $|\sin 2x| + |\cos 2x| = |\sin y|$ is

(a) 1 (b) 2 (c) 3 (d) 4

11. Let $\theta \in [0, 4\pi]$ satisfying the equation $(\sin\theta + 2)(\sin\theta + 3)(\sin\theta + 4) = 6$. If the sum of all values of θ is $K\pi$, then value of K is

(a) 6 (b) 5 (c) 4 (d) 2

12. The line $x + y = 1$ meets x-axis at A and y-axis at B. P is the mid-point of AB. P_1 is the foot of the perpendicular from P to OA; $M_1$is that from P_1 to OP; P_2 is that from M_1 to OA; M_2 is that from P_2 to OP; P_3 is that from M_2 to OA and so on. If P_n denotes the n^{th} foot of the perpendicular on OA from M_{n-1}, then OP_n is

(a) $\frac{1}{2}$ (b) $\frac{1}{2^n}$ (c) $\frac{1}{2^{n/2}}$ (d) $\frac{1}{\sqrt{2}}$

13. The orthocentre of the triangle formed by the lines $x + y = 1$, $2x + 3y = 6$ and $4x - y + 4 = 0$ lies in

(a) I quadrant (b) II quadrant

(c) III quadrant (d) IV quadrant

14. If $f : R \to R$ defined by $f(x) = x|x|$, then f is

(a) one-one but not onto

(b) onto but not one-one

(c) both one-one and onto

(d) neither one-one nor onto

15. The domain of the function

$$f(x)=\sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right) \text{ is}$$

(a) $[-2, 2]$ (b) $[-2, -1]$

(c) $[1, 2]$ (d) $[-2, -1] \cup [1, 2]$

16. $\lim\limits_{x\to 0}\dfrac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}$ is equal to $(a > 0)$

(a) $\log_e a$ (b) 1 (c) 0 (d) ∞

17. Let $f: R \to R$ be a differentiable function satisfying $f(y)f(x-y)=f(x)\ \forall x, y \in R$ and $f'(0) = p$, $f'(5) = q$, then $f(5)$ is

(a) p^2/q (b) p/q (c) q/p (d) q

18. $f(x)=\begin{cases}\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{\frac{1}{x}}, & \text{if } x\neq 0\\ \lambda, & \text{if } x=0\end{cases}$ is continuous at $x = 0$, then value of λ is

(a) 1 (b) e (c) e^2 (d) 0

19. If $2p^2 - 3q^2 + 4pq - p = 0$ and a variable line $px + qy = 1$ always touches a parabola whose axis is parallel to X-axis, then equation of the parabola is

(a) $(y - 4)^2 = 24(x - 2)$ (b) $(y - 3)^2 = 12(x - 1)$

(c) $(y - 4)^2 = 12(x - 2)$ (d) $(y - 2)^3 = 24(x - 4)$

20. The locus of point of intersection of tangents to the parabola $y^2 = 4ax$, the angle between them being always 45°, is

(a) $x^2 - y^2 + 6ax - a^2 = 0$

(b) $x^2 - y^2 - 6ax + a^2 = 0$

(c) $x^2 - y^2 + 6ax + a^2 = 0$

(d) $x^2 - y^2 - 6ax - a^2 = 0$

NUMERICAL VALUE TYPE

21. The ratio of sum of first three terms of a G.P. to the sum of first six terms is 64 : 91 ,the common ratio of G.P. is _____.

22. If $1, \log_{81}(3^x+48), \log_9\left(3^x-\frac{8}{3}\right)$ are in A.P., then the value of x equals _____.

23. If α, β are the roots of the equation $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ (where $n \geq 2$), then $aS_{n+1} + bS_n + cS_{n-1}$ = _____.

24. The number of positive integral solutions of the equation $\begin{vmatrix} y^3+1 & y^2z & y^2x \\ yz^2 & z^3+1 & z^2x \\ yx^2 & x^2z & x^3+1 \end{vmatrix}=11$ is _____.

25. If $a - 2b + c = 1$, then the value of $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ is _____.

SOLUTIONS

1. (a) : $\tan(A+B)=\dfrac{P}{1-Q}$

Then $\sin^2(A+B)=\dfrac{\tan^2(A+B)}{1+\tan^2(A+B)}=\dfrac{P^2}{P^2+(1-Q)^2}$

2. (b) : $\left(\sum_{k=1}^{n} a_k\right)^2=\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+.....+\frac{1}{n}-\frac{1}{n+1}\right)^2$

$=\dfrac{n^2}{(n+1)^2}$

3. (d) : $\sum_{k=1}^{\infty}\frac{1}{k!}(1+2+2^2+.....+2^{k-1})$

$=\sum_{k=1}^{\infty}\frac{2^k-1}{k!}=e^2-e$

4. (a) : The general term of the series is $t_n=\dfrac{x^{2^{n-1}}}{1-x^{2^n}}$

$=\dfrac{1+x^{2^{n-1}}-1}{\left(1+x^{2^{n-1}}\right)\left(1-x^{2^{n-1}}\right)}$

$\therefore \quad t_n=\dfrac{1}{1-x^{2^{n-1}}}-\dfrac{1}{1-x^{2^n}}$

Now, $S_n=\sum_{n=1}^{n} t_n=\left[\left\{\frac{1}{1-x}-\frac{1}{1-x^2}\right\}\right.$

$+\left\{\frac{1}{1-x^2}-\frac{1}{1-x^4}\right\}+....+\left.\left\{\frac{1}{1-x^{2^{n-1}}}-\frac{1}{1-x^{2^n}}\right\}\right]$

$=\dfrac{1}{1-x}-\dfrac{1}{1-x^{2^n}}$

$\therefore$ The sum to infinite terms

$=\lim\limits_{n\to\infty} S_n=\dfrac{1}{1-x}-1=\dfrac{x}{1-x}\left[\because \lim\limits_{n\to\infty} x^{2^n}=0, \text{ as } |x|<1\right]$

5. (b) : We have, $a=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}=\omega^2$

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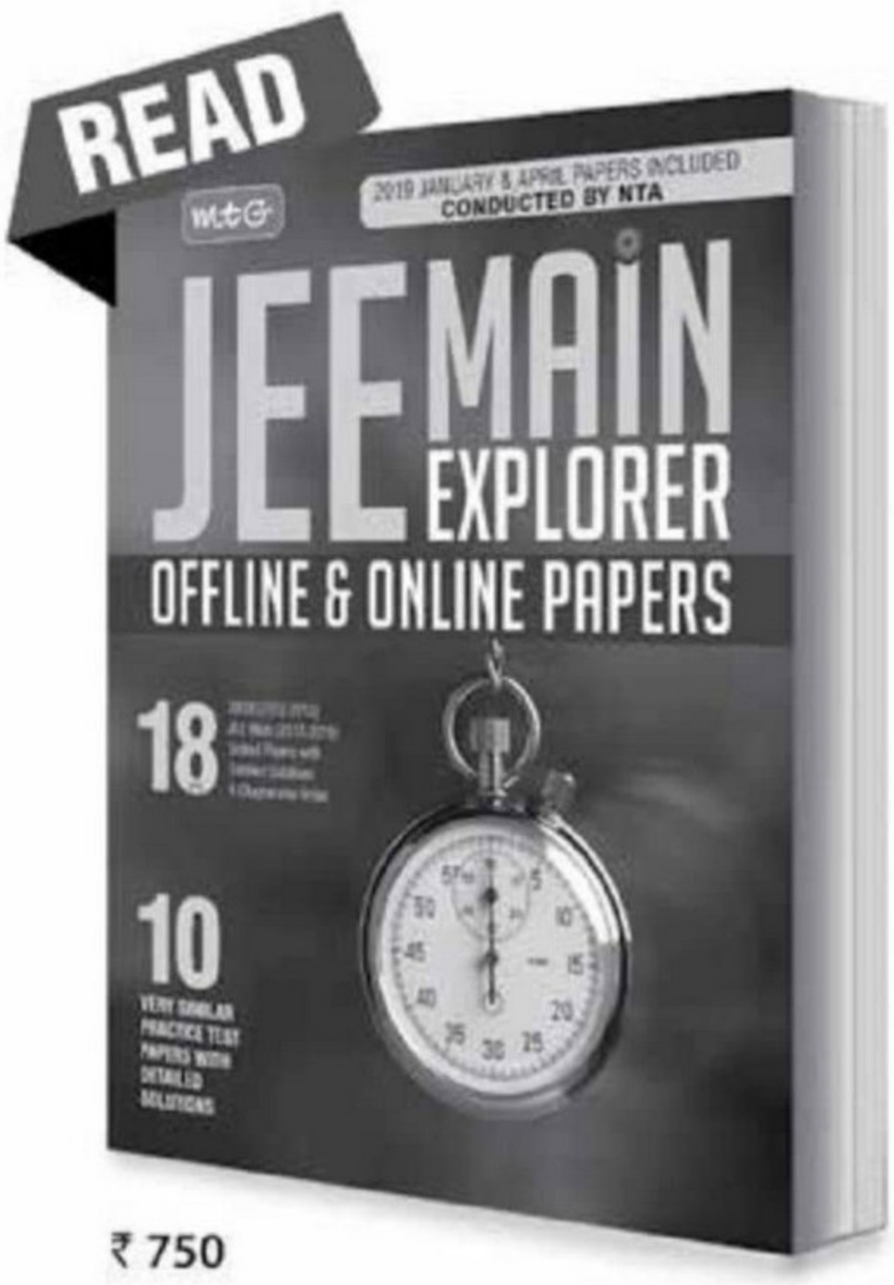

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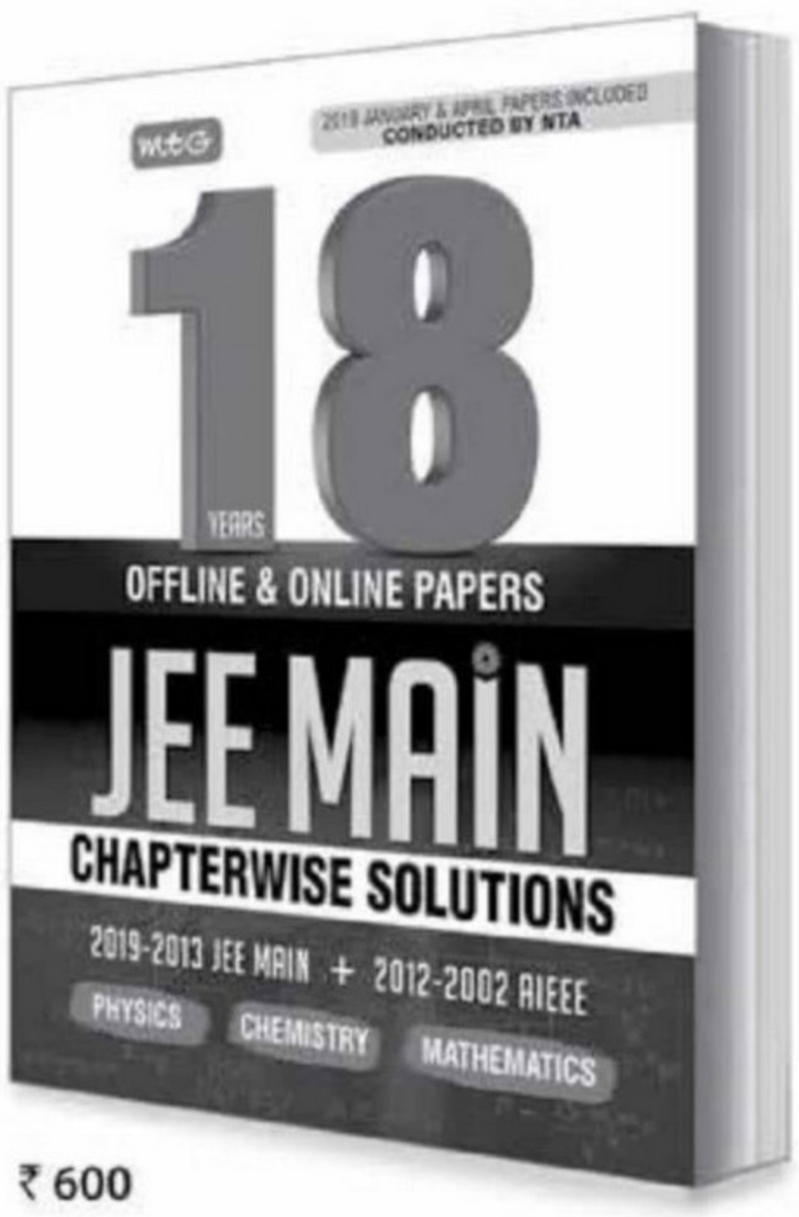

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$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{vmatrix} = 3(\omega - \omega^2), \text{which is purely imaginary.}$$

6. (a) :

$$\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix} = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$$

[Applying $C_2 \to C_2 + C_1$, $C_3 \to C_3 + C_2$]

$$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix} \text{ [Applying } C_3 \to C_3 + C_2]$$

$\therefore$ Required result = 0

7. (c) : $(A^{-1}BA)^2 = (A^{-1}BA)(A^{-1}BA)$

$= A^{-1}B(AA^{-1})BA = A^{-1}BIBA = A^{-1}B^2A$

and $(A^{-1}BA)^3 = (A^{-1}B^2A)(A^{-1}BA) = A^{-1}B^2(AA^{-1})BA$

$= A^{-1}B^2IBA = A^{-1}B^3A$ and so on

$\Rightarrow (A^{-1}BA)^n = A^{-1}B^nA$

8. (a) : We have, $A^3 = B^3$ ...(i)

and $A^2B = B^2A$ ...(ii)

Subtracting (ii) from (i), we get

$(A^2 + B^2)(A - B) = O$

$\Rightarrow A^2 + B^2 = O \quad (\because |A - B| \neq 0)$

9. (b) : $x^2 + y^2 = 3 + 2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right)$

$= 3 + 2\left(\frac{-1}{2}\right) = 2$

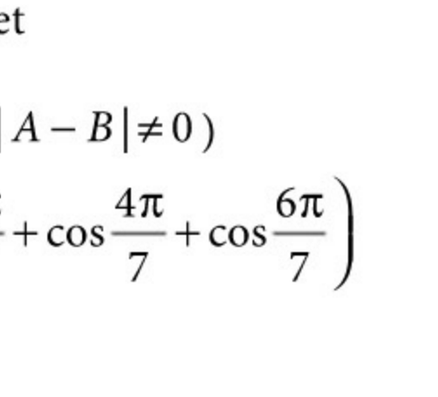
10. (d) : $1 \le |\sin 2x| + |\cos 2x| \le \sqrt{2}$

and $|\sin y| \le 1 \Rightarrow \sin y = \pm 1 \Rightarrow y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

11. (b) : $\sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}, \frac{7\pi}{2}$

$\therefore K\pi = 5\pi \Rightarrow K = 5$

12. (b) : $x + y = 1$ meets x-axis at $A(1, 0)$ and y-axis at $B(0, 1)$. The coordinates of P are $(1/2, 1/2)$ and PP_1 is perpendicular to OA.



$\Rightarrow OP_1 = P_1P = 1/2$

Equation of line OP is $y = x$

We have

$(OM_{n-1})^2 = (OP_n)^2 + (P_nM_{n-1})^2 = 2(OP_n)^2 = 2p_n^2$ (say)

Also, $(OP_{n-1})^2 = (OM_{n-1})^2 + (P_{n-1}M_{n-1})^2$

$\Rightarrow (OP_{n-1})^2 = 2p_n^2 + \frac{1}{2}p_{n-1}^2$

$\Rightarrow p_n^2 = \frac{1}{4}p_{n-1}^2 \Rightarrow p_n = \frac{1}{2}p_{n-1}$

$\therefore OP_n = p_n = \frac{1}{2}p_{n-1} = \frac{1}{2^2}p_{n-2} = ... = \frac{1}{2^{n-1}}p_1 = \frac{1}{2^n}$

13. (a) : Coordinates of A and B are $(-3, 4)$ and $\left(-\frac{3}{5}, \frac{8}{5}\right)$ respectively. Let orthocentre is $P(h, k)$.

Then, (slope of PA) $\times$ (slope of BC) $= -1$

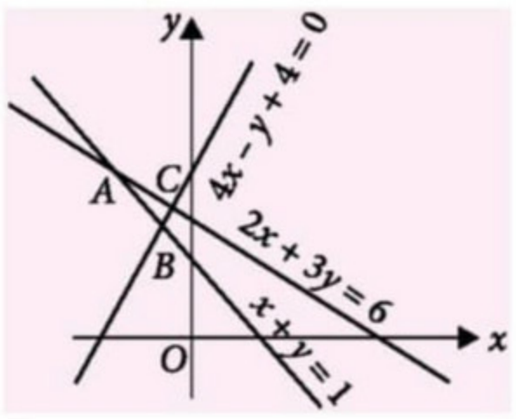
$\Rightarrow \frac{k-4}{h+3} \times 4 = -1$

$\Rightarrow 4k - 16 = -h - 3$

$\Rightarrow h + 4k = 13$ ...(i)

and slope of $PB \times$ slope of $AC = -1$

$\Rightarrow \frac{k - \frac{8}{5}}{h + \frac{3}{5}} \times \left(-\frac{2}{3}\right) = -1$



$\Rightarrow 10k - 16 = 15h + 9$

$\Rightarrow 15h - 10k + 25 = 0 \Rightarrow 3h - 2k + 5 = 0$ ... (ii)

Solving (i) and (ii), we get $h = \frac{3}{7}, k = \frac{22}{7}$

Hence, orthocentre lies in I quadrant.

14. (c) : Given that $f(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

Clearly, $f(x)$ is both one-one and onto.

15. (d) : $f(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right) \in \mathbf{R}$

$\Leftrightarrow -1 \le \log_2\left(\frac{x^2}{2}\right) \le 1 \Leftrightarrow \frac{1}{2} \le \frac{x^2}{2} \le 2 \Leftrightarrow 1 \le x^2 \le 4$

$\Leftrightarrow x \in [-2, -1] \cup [1, 2]$

16. (a) : We have,

$$\lim_{x\to 0} \frac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x} = \lim_{x\to 0} a^{\sin x}\left(\frac{a^{\tan x - \sin x} - 1}{\tan x - \sin x}\right)$$

Monthly Test Drive-3 CLASS XI ANSWER KEY

1. (b)	2. (c)	3. (a)	4. (a)	5. (a)
6. (c)	7. (a)	8. (b, c)	9. (a, b)	10. (a, b)
11. (a, c, d)	12. (a, c)	13. (a, b)	14. (b)	15. (d)
16. (b)	17. (5)	18. (6)	19. (3)	20. (8)

$$= \lim_{x\to 0}(a^{\sin x}) \times \lim_{t\to 0}\left(\frac{a^t - 1}{t}\right) (\text{where } t = \tan x - \sin x)$$

$$= a^0 \times \log_e a = \log_e a$$

17. (c) : $y = 0 \Rightarrow f(0) = 1$ and $x = 0 \Rightarrow f(-y) = \dfrac{1}{f(y)}$

Hence, $f(x + y) = f(x)f(y)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= f(x)\lim_{h\to 0}\frac{f(h)-1}{h} = f(x).f'(0) = pf(x) \qquad \ldots(i)$$

Put $x = 5$ in (i), we get $f(5) = \dfrac{q}{p}$

18. (c) : We have, $\lambda = \lim_{x\to 0}\left(\dfrac{1+\tan x}{1-\tan x}\right)^{\frac{1}{x}} = e^2$

19. (c) : The parabola be $(y - a)^2 = 4b\,(x - c)$

Equation of tangent is $(y-a) = -\dfrac{p}{q}(x-c) - \dfrac{bq}{p}$

Comparing with $px + qy = 1$, we get

$cp^2 - bq^2 + apq - p = 0$

$$\therefore \quad \frac{c}{2} = \frac{b}{3} = \frac{a}{4} = 1$$

Thus, equation of parabola is $(y - 4)^2 = 12(x - 2)$

20. (c) : Equation of tangent is $y = mx + \dfrac{a}{m}$

$$\Rightarrow m^2x - my + a = 0 \Rightarrow m_1 + m_2 = \frac{y}{x}, m_1m_2 = \frac{a}{x}$$

$$\tan 45° = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| \Rightarrow \left(\frac{y}{x}\right)^2 - 4\left(\frac{a}{x}\right) = \left(1 + \frac{a}{x}\right)^2$$

$\Rightarrow \quad x^2 - y^2 + 6ax + a^2 = 0$

21. (0.75) : Given $\dfrac{S_3}{S_6} = \dfrac{64}{91} = \dfrac{a(r^3-1)}{a(r^6-1)}$

$$\Rightarrow \quad \frac{(r^3-1)}{(r^3+1)(r^3-1)} = \frac{64}{91} \quad \Rightarrow \quad r^3 = \frac{27}{64} \quad \therefore \quad r = \frac{3}{4}$$

22. (2) : Given $1, \log_{9^2}(3^x + 48), \log_9(3^x - 8/3) \in$ A.P.

$$\Rightarrow \log_9 9, \frac{1}{2}\log_9(3^x + 48), \log_9(3^x - 8/3) \in \text{A.P.}$$

$\Rightarrow$ $9, (3^x + 48)^{1/2}, 3^x - 8/3 \in$ G.P.

[$\because$ $\log a, \log b, \log c \in$ A.P. $\Rightarrow$ $a, b, c \in$ G.P.]

$\therefore$ $3^x + 48 = 9(3^x - 8/3)$

$\Rightarrow$ $8 \cdot 3^x = 72 \Rightarrow 3^x = 9 \Rightarrow 3^x = 3^2 \Rightarrow x = 2.$

23. (0) : $S_{n+1} = \alpha^{n+1} + \beta^{n+1}$

$$= (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1}) = -\frac{b}{a}\cdot S_n - \frac{c}{a}\cdot S_{n-1}$$

24. (3) : Multiply by y, z and x in rows 1,2 and 3 respectively and then take common y, z and x from column 1, 2 and 3 respectively, we get

$$\begin{vmatrix} y^3+1 & y^3 & y^3 \\ z^3 & z^3+1 & z^3 \\ x^3 & x^3 & x^3+1 \end{vmatrix} = 11$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & y^3 \\ -1 & 1 & z^3 \\ 0 & -1 & x^3+1 \end{vmatrix} = 11$$

(Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$)

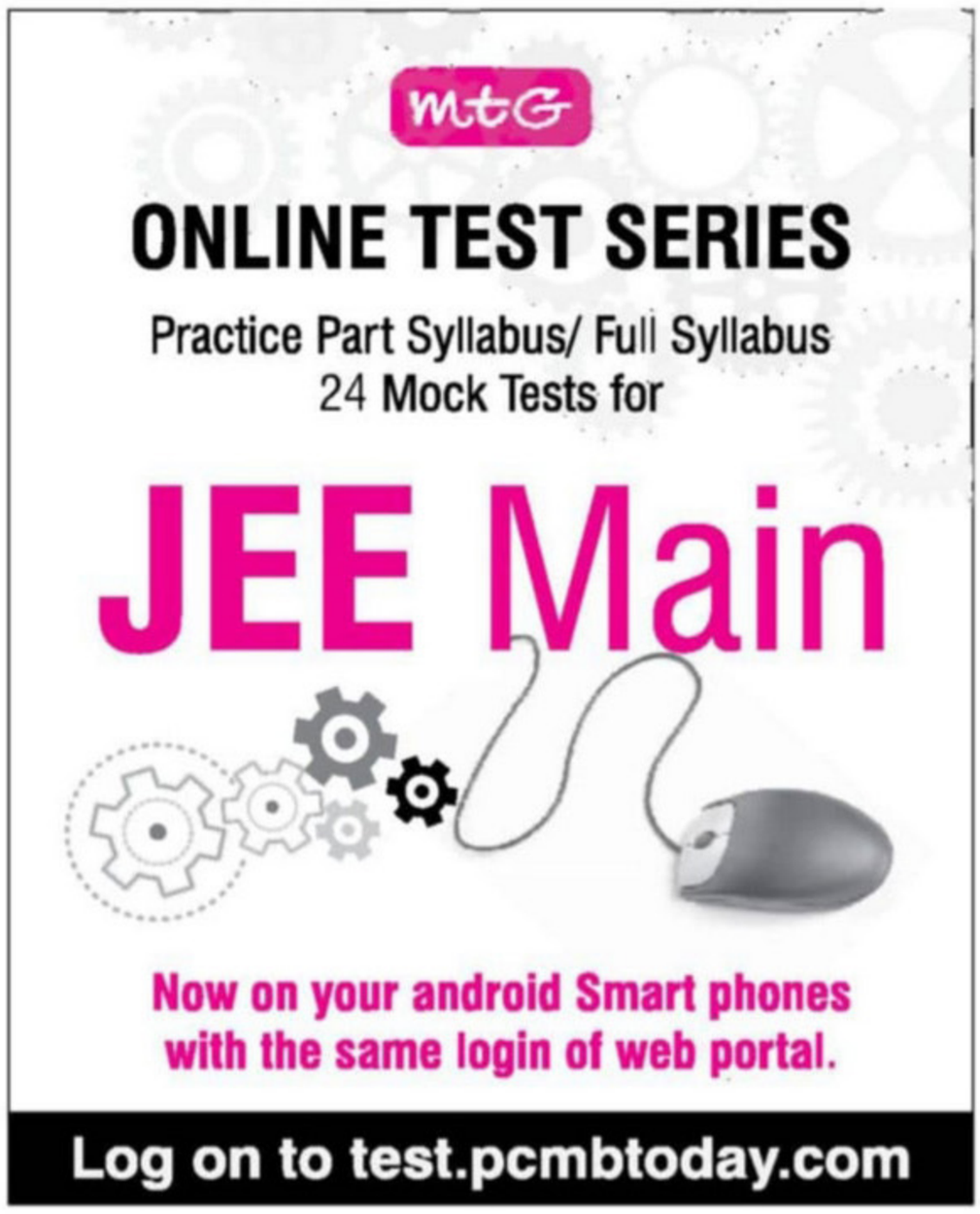
$\Rightarrow$ $1(x^3 + 1 + z^3) + y^3(1) = 11$

$\Rightarrow$ $x^3 + y^3 + z^3 = 10$

So, solutions are (1, 1, 2), (1, 2, 1) or (2, 1, 1)

25. (–1) : Applying the operation, $R_1 \to R_1 - 2R_2 + R_3$ and $R_3 \to R_3 - R_2$ and using $a - 2b + c = 1$, the determinant reduces to

$$\begin{vmatrix} 0 & 0 & 1 \\ x+2 & x+3 & x+b \\ 1 & 1 & c-b \end{vmatrix} = -1$$



JEE 2020 ADVANCED

Exam on 27th September

PRACTICE PAPER

PAPER-1

SECTION-1

SINGLE CORRECT ANSWER TYPE

1. A committee of 6 is chosen from 10 men and 7 women so as to contain atleast 3 men and 2 women. The number of ways this can be done, if two particular women refuse to serve on the same committee, is

(a) 8000 (b) 7800 (c) 7600 (d) 7200

2. If $\ln_{\sin x} \cos x + \ln_{\cos x} \sin x = 2$, then $x =$

(a) $\frac{n\pi}{4}$ (b) $(n+1)\frac{\pi}{4}$

(c) $(2n+1)\frac{\pi}{4}$ (d) $(4n+1)\frac{\pi}{4}$

3. The slope of the straight line which is both tangent and normal to the curve $4x^3 = 27y^2$ is

(a) ± 1 (b) $\pm\frac{1}{2}$ (c) $\pm\frac{1}{\sqrt{2}}$ (d) $\pm\sqrt{2}$

4. If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of the reciprocals of their squares, then $\frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

5. A person goes to office either by car, scooter, bus or train the probabilities of which being $\frac{1}{7}, \frac{3}{7}, \frac{2}{7}, \frac{1}{7}$ respectively. The probability that he reaches office late, if he takes car, scooter, bus or train is $\frac{2}{9}, \frac{1}{9}, \frac{4}{9}, \frac{1}{9}$ respectively. If he reached office in time, the probability that he travelled by car is

(a) $\frac{1}{5}$ (b) $\frac{1}{9}$ (c) $\frac{2}{11}$ (d) $\frac{1}{7}$

SECTION-2

MORE THAN ONE CORRECT ANSWER TYPE

6. Let $f(x) = \ln|x|$ and $g(x) = \sin x$. If A is the range of $f(g(x))$ and B is the range of $g(f(x))$, then

(a) $A \cup B = (-\infty, 1]$ (b) $A \cup B = (-\infty, \infty)$
(c) $A \cap B = [-1, 0]$ (d) $A \cap B = [0, 1]$

7. Let $y = f(x)$ be a curve in the first quadrant such that the triangle formed by the co-ordinate axes and the tangent at any point on the curve has area 2. If $y(1) = 1$, then $y(2) =$

(a) 0 (b) 1 (c) 2 (d) 1/2

8. If $\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$ then $\theta =$

(a) $\frac{7\pi}{24}$ (b) $\frac{5\pi}{24}$ (c) $\frac{11\pi}{24}$ (d) $\frac{\pi}{24}$

9. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0, y_2 < 0$ be the ends of the latusrectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latusrectum PQ are

(a) $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$

(b) $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$

(c) $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$

(d) $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

10. The equation of the plane containing the line $2x - y + z - 3 = 0, 3x + y + z = 5$ and at a distance of $\frac{1}{\sqrt{6}}$ from the point $(2, 1, -1)$ is

(a) $x + y + z - 3 = 0$ (b) $2x - y - z - 3 = 0$
(c) $2x - y + z - 3 = 0$
(d) $62x + 29y + 19z - 105 = 0$

11. Tangents are drawn from a point P on the circle $C: x^2 + y^2 = a^2$ to the circle $C_1: x^2 + y^2 = b^2$. If these tangents cut the circle C at Q and R and if QR is a tangent to the circle C_1, then the area of triangle PQR is

(a) $3\sqrt{3}\,b^2$ (b) $\frac{3\sqrt{3}}{4}a^2$

(c) $\frac{3\sqrt{3}}{2}ab$ (d) $2\sqrt{3}\,ab$

12. The internal bisector of $\angle A$ of triangle ABC meets side BC at D. A line drawn through D perpendicular to AD meets the side AC at E and the side AB at F. If a, b, c are the sides of triangle ABC, then

(a) AE is H.M. of b and c

(b) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$ (c) $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$

(d) triangle AEF is isosceles.

13. Let a, b, c, d, e be five numbers such that a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in H.P. If $a = 2$ and $e = 18$, then $b =$

(a) 2 (b) -2 (c) 4 (d) -4

SECTION-3

INTEGER ANSWER TYPE

14. If $n \in N$, then the highest integer m such that 2^m divides $3^{2n+2} - 8n - 9$ is

15. If $\lim_{x\to 0}\left(1 + x + \frac{f(x)}{x}\right)^{1/x} = e^3$, then $\lim_{x\to 0}\frac{f(x)}{x^2} =$

16. Let $px^4 + qx^3 + rx^2 + sx + t = \begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & -2x & x-4 \\ x-3 & x+4 & 3x \end{vmatrix}$ be an identity, where p, q, r, s, t are constants. Then $t =$

17. Let $n \in N$, $n \le 5$. If $I_n = \int_0^1 e^x (x-1)^n dx = 16 - 6e$, then $n =$

18. If ω is complex cube root of unity and a, b, c are such that $\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} = 2\omega^2$ and $\frac{1}{a+\omega^2} + \frac{1}{b+\omega^2} + \frac{1}{c+\omega^2} = 2\omega$, then $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} =$

PAPER-2

SECTION-1

SINGLE CORRECT ANSWER TYPE

1. A plane moves such that the volume of the tetrahedron formed with coordinate planes is a constant k. The locus of the foot of perpendicular from the origin O on it is $(x^2 + y^2 + z^2)^3 =$

(a) $12\,kxyz$ (b) $24\,kxyz$

(c) $48\,kxyz$ (d) $6\,kxyz$

2. An open box with a square base is to be made from 400 m^2 of lumber. When its volume is maximum, the ratio of an edge of the base to the height is

(a) 1 (b) $\frac{1}{2}$ (c) 2 (d) $\sqrt{2}$

3. $S = \sum_{r=1}^{n-1}\cos^2\frac{r\pi}{n} =$

(a) $\frac{n}{2}$ (b) $\frac{n-1}{2}$ (c) $\frac{n-2}{2}$ (d) $\frac{n+1}{2}$

4. Let S be the sum, P be the product and R be the sum of reciprocals of n terms of a G.P. Then

(a) $R = S \cdot P^{1/n}$ (b) $R = S \cdot P^{2/n}$

(c) $S = R \cdot P^{1/n}$ (d) $S = R \cdot P^{2/n}$

5. Let $f(x)$ be a function such that $f'(x) = f(x), f(0) = 1$ and $g(x)$ be a function such that $f(x) + g(x) = x^2$. Then $\int_0^1 f(x)g(x)dx =$

(a) $e - \frac{e^2}{2} - \frac{5}{2}$ (b) $e + \frac{e^2}{2} - \frac{3}{2}$

(c) $e - \frac{e^2}{2} - \frac{3}{2}$ (d) $e + \frac{e^2}{2} + \frac{5}{2}$

6. If the system of equations $x + ky + 3z = 0$, $3x + ky - 2z = 0$, $2x + 3y - 4z = 0$ has non trivial solution, then $\frac{xy}{z^2} =$

(a) $\frac{5}{6}$ (b) $\frac{-5}{6}$ (c) $\frac{6}{5}$ (d) $\frac{-6}{5}$

SECTION-2

MORE THAN ONE CORRECT ANSWER TYPE

7. If $x + |y| = 2y$, then y as a function of x is

(a) defined for all x

(b) continuous at $x = 0$

(c) differentiable for all x

(d) $\frac{dy}{dx} = \frac{1}{3}$ for $x < 0$

8. Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors, equally inclined to each other at angle θ, where $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices of a triangle and $\vec{g}$ is the position vector of its centroid, then

(a) $|\vec{g}| \ge \frac{1}{\sqrt{3}}$ (b) $|\vec{g}| \le 1$

(c) $|\vec{g}| \le \sqrt{\frac{2}{3}}$ (d) $|\vec{g}| \ge 1$

9. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t) z_1 + tz_2$, for some real number t with $0 < t < 1$, then

(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(b) $\arg(z - z_1) = \arg|z - z_2|$

(c) $\arg(z - z_1) = \arg(z_2 - z_1)$

(d) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

10. If a hyperbola passes through a focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and its transverse and conjugate axes coincide with major and minor axes of the ellipse, and the product of their eccentricities is 1, then

(a) the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

(b) the hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$

(c) a focus of hyperbola is (5, 0)

(d) a focus of hyperbola is $(5\sqrt{3}, 0)$

11. Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$

and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}$. Then

(a) $S_n < \frac{\pi}{3\sqrt{3}}$ (b) $S_n > \frac{\pi}{3\sqrt{3}}$

(c) $T_n < \frac{\pi}{3\sqrt{3}}$ (d) $T_n > \frac{\pi}{3\sqrt{3}}$

12. The area of the triangle formed by the tangent to the curve $y = x^2 + bx - b$ at the point (1, 1) and the coordinate axes is 2. Then $b =$

(a) 3 (b) -3

(c) $1 + 2\sqrt{2}$ (d) $1 - 2\sqrt{2}$

13. If the equations $x + y = 1$, $(c + 2)x + (c + 4)y = 6$, $(c + 2)^2 x + (c + 4)^2 y = 36$ are consistent, then $c =$

(a) 1 (b) 2 (c) 3 (d) 4

14. If the solution of $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = x$, $y(0) = y(1) = 1$ is given by $y^2 = f(x)$ then

(a) $f(x)$ is monotonically increasing $\forall\ x \in (1, \infty)$

(b) $f(x) = 0$ has only one root

(c) $f(x)$ is neither even nor odd

(d) $f(x)$ has 3 real roots

SECTION-3

COMPREHENSION TYPE

Paragraph for Q. No. 15 and 16

For $x \in \left(0, \frac{\pi}{4}\right)$,

let $S_n = \sum_{r=1}^{2n} \sin(\sin^{-1} x^{3r-2})$, $C_n = \sum_{r=1}^{2n} \cos(\cos^{-1} x^{3r-1})$

and $T_n = \sum_{r=1}^{2n} \tan(\tan^{-1} x^{3r})$, where $n \in N$ and $n \ge 3$

15. The correct order of S_n, C_n and T_n is given by

(a) $S_n > T_n > C_n$ (b) $S_n < C_n < T_n$

(c) $S_n < T_n < C_n$ (d) $S_n > C_n > T_n$

16. The value of 'x' for which $S_n = C_n + T_n$, is

(a) $\sin\left(\frac{\pi}{5}\right)$ (b) $2\sin\left(\frac{\pi}{5}\right)$

(c) $2\sin\left(\frac{\pi}{10}\right)$ (d) $\sin\left(\frac{\pi}{10}\right)$

Paragraph for Q. No. 17 and 18

The normal at any point (x_1, y_1) of curve is line perpendicular to tangent at (x_1, y_1). In case of parabola $y^2 = 4ax$ the equation of normal is $y = mx - 2am - am^2$ (m is slope of normal). In case of rectangular hyperbola $xy = c^2$ the equation of normal at $\left(ct, \frac{c}{t}\right)$ is $xt^3 - yt - ct^4 + c = 0$ and the shortest distance between any two curve always exist along the common normal.

17. If normal at (5, 3) of hyperbola $xy - y - 2x - 2 = 0$ intersects the curve again at $(\alpha, \beta - 29)$, then $\frac{\beta}{\alpha}$ is

(a) 10 (b) 20 (c) 30 (d) 40

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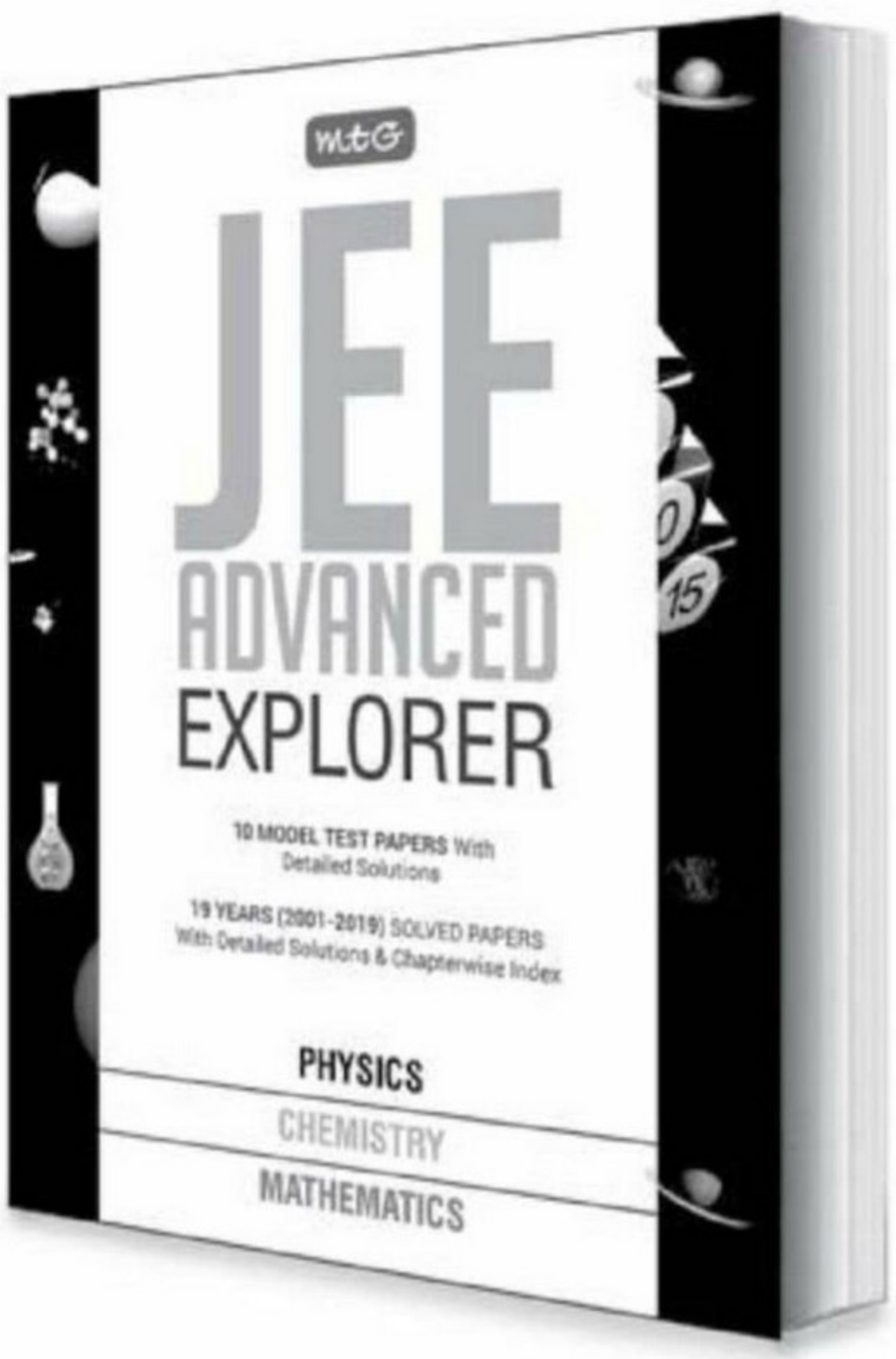

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18. If the shortest distance between $2y^2 - 2x + 1 = 0$ and $2x^2 - 2y + 1 = 0$ is d, then the number of solutions of $|\sin\alpha| = 2\sqrt{2}d$ $(\alpha \in [-\pi, 2\pi])$ is

(a) 3 (b) 4

(c) 5 (d) none of these

SOLUTIONS

PAPER-1

1. (b) : Let the men be $M_1, M_2, ..., M_{10}$ and women be $W_1, W_2, ..., W_7$. Let W_1 and W_2 do not want to be on the same committee.

The six member committee can contain 4 men and 2 women or 3 men and 3 women.

The number of ways of forming $4M$, $2W$ committee is

$$\binom{10}{4}\cdot\left[\binom{5}{2}+2\binom{5}{1}\right] = 4200 \qquad ...(1)$$

$\binom{5}{2}$ is the number of ways without W_1 and W_2

$\binom{5}{1}$ is the number of ways with W_1, and without W_2,

or with W_2 and without W_1. The number of ways of forming $3M$, $3W$ committee is

$$\binom{10}{3}\cdot\left[\binom{5}{3}+2\binom{5}{2}\right] = 3600 \qquad ...(2)$$

$\binom{5}{3}$ is the number of ways without W_1 and W_2. $\binom{5}{2}$ is the number of ways with W_1 or W_2 but not both.

$\therefore$ The desired number of ways is 4200 + 3600 = 7800.

2. (d) : $\dfrac{\ln\cos x}{\ln\sin x} + \dfrac{\ln\sin x}{\ln\cos x} = 2$

Since, $t + \dfrac{1}{t} = 2 \Rightarrow t = 1$

$$\Rightarrow \frac{\ln\sin x}{\ln\cos x} = 1 \Rightarrow \ln\sin x - \ln\cos x = 0$$

$$\Rightarrow \ln\tan x = 0 \Rightarrow \tan x = 1 = \tan\frac{\pi}{4}$$

$$\therefore \quad x = n\pi + \frac{\pi}{4} = (4n+1)\frac{\pi}{4}.$$

3. (d) : $x = 3t^2, y = 2t^3, \dfrac{dy}{dx} = t$

The tangent at t, $y - 2t^3 = t(x - 3t^2)$, $tx - y = t^3$...(1)

The normal at t_1, $t_1 y + x = 2t_1^4 + 3t_1^2$ ...(2)

(1), (2) are identical, $\dfrac{t}{1} = \dfrac{-1}{t_1} = \dfrac{t^3}{2t_1^4 + 3t_1^2}$

$$\Rightarrow -t^3 = 2t_1^3 + 3t_1,\ t_1 = -\frac{1}{t}.$$

Eliminating t_1, we get $t^6 = 2 + 3t^2$

$$\Rightarrow t^2 = 2,\ t = \pm\sqrt{2}.$$

4. (c): $\alpha + \beta = \dfrac{-b}{a}, \alpha\beta = \dfrac{c}{a}, \alpha + \beta = \dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$

$$\Rightarrow \quad \alpha + \beta = \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$\therefore \quad (\alpha + \beta)(\alpha\beta)^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\Rightarrow \quad -\frac{b}{a}\cdot\frac{c^2}{a^2} = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ca}{a^2}$$

$$\Rightarrow -bc^2 = ab^2 - 2ca^2 \Rightarrow ab^2 + bc^2 = 2ca^2$$

$\Rightarrow$ ab^2, ca^2, bc^2 are in A.P.

Dividing by abc, we get $\dfrac{b}{c}, \dfrac{a}{b}, \dfrac{c}{a}$ are in A.P. and $\dfrac{c}{a}, \dfrac{a}{b}, \dfrac{b}{c}$ are in A.P. so, $\dfrac{a}{c}, \dfrac{b}{a}, \dfrac{c}{b}$ are in H.P.

5. (d) : Let C, S, B, T be the events of the person going by car, scooter, bus, train respectively.

$$P(C) = \frac{1}{7}, P(S) = \frac{3}{7}, P(B) = \frac{2}{7}, P(T) = \frac{1}{7}.$$

Let $\bar{L}$ be the event that the person reaching the office in time.

$$P(\bar{L}|C) = \frac{7}{9}, P(\bar{L}|S) = \frac{8}{9}, P(\bar{L}|B) = \frac{5}{9}, P(\bar{L}|T) = \frac{8}{9}$$

$$\therefore \quad P(C|\bar{L}) = \frac{\frac{1}{7}\cdot\frac{7}{9}}{\frac{1}{7}\cdot\frac{7}{9} + \frac{3}{7}\cdot\frac{8}{9} + \frac{2}{7}\cdot\frac{5}{9} + \frac{1}{7}\cdot\frac{8}{9}} = \frac{7}{49} = \frac{1}{7}$$

6. (a, c) : The range of $|\sin x| = [0, 1]$.

A = range of $\ln|\sin x| = (-\infty, 0]$.

The range of $\ln|x| = (-\infty, \infty)$.

B = range of $\sin\ln|x| = [-1, 1]$

$\therefore$ $A \cup B = (-\infty, 0] \cup [-1, 1] = (-\infty, 1]$,

$A \cap B = (-\infty, 0] \cap [-1, 1] = [-1, 0]$.

7. (a, d) : The tangent at $P(x, y)$ is $Y - y - (X - x)y_1 = 0$

It meets x-axis at $A\left(x - \dfrac{y}{y_1}, 0\right)$ and y-axis at $B(0, y - xy_1)$.

Area of $\Delta OAB = 2 \Rightarrow OA \cdot OB = 4$

$$\Rightarrow \left(x - \frac{y}{p}\right)(y - xp) = 4, \text{ where } p = \frac{dy}{dx}$$

$\Rightarrow\ (y - xp)^2 = -4p$

$\Rightarrow\ y - xp = 2\sqrt{-p}\ \Rightarrow\ y = xp + 2\sqrt{-p}$

The general solution is $y = cx + 2\sqrt{-c}$ ...(1)

$x = 1, y = 1 \Rightarrow c = -1, y = 2 - x, y(2) = 0$

Differentiating (1) w.r.t. c, $0 = x - \frac{1}{\sqrt{-c}}$ ...(2)

Eliminating c from (1), using (2)

$$y = -\frac{1}{x} + \frac{2}{x} = \frac{1}{x}, y(2) = \frac{1}{2}$$

8. (a, c) : Subtracting R_3 from R_1 and R_2,

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \sin^2\theta & \cos^2\theta & 1 + 4\sin 4\theta \end{vmatrix} = 0$$

$\Rightarrow\ \sin^2\theta + \cos^2\theta + 1 + 4\sin 4\theta = 0$

$\Rightarrow\ \sin 4\theta = -\frac{1}{2}\ \Rightarrow\ \theta = \frac{7\pi}{24}, \frac{11\pi}{24}$

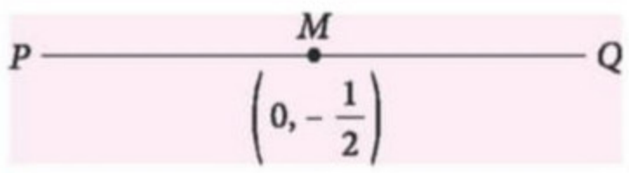
9. (b, c) : $\frac{x^2}{4} + \frac{y^2}{1} = 1$, here, $a = 2, b = 1$

The ends of latusrectum PQ are

$$\left(\pm\sqrt{a^2 - b^2}, -\frac{b^2}{a}\right)$$

So, $P = \left(-\sqrt{3}, -\frac{1}{2}\right), Q = \left(\sqrt{3}, -\frac{1}{2}\right)$

The midpoint of PQ is $M\left(0, -\frac{1}{2}\right)$, which is the focus of the parabola.



The vertex of the parabola is $V\left(0, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)$.

$\therefore$ The parabolas are $x^2 = \mp 2\sqrt{3}\left(y - \left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)\right)$

i.e., $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$ and $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$.

10. (c, d) : The plane is

$2x - y + z - 3 + \lambda(3x + y + z - 5) = 0$ ...(i)

or $(2 + 3\lambda)x + (\lambda - 1)y + (\lambda + 1)z - (5\lambda + 3) = 0$

Its distance from $(2, 1, -1)$ is $\frac{1}{\sqrt{6}}$

$$\frac{|4 + 6\lambda + \lambda - 1 - \lambda - 1 - 5\lambda - 3|}{\sqrt{(2 + 3\lambda)^2 + (\lambda - 1)^2 + (\lambda + 1)^2}} = \frac{1}{\sqrt{6}}$$

$\Rightarrow\ 6(\lambda - 1)^2 = 11\lambda^2 + 12\lambda + 6\ \Rightarrow\ (5\lambda + 24)\lambda = 0$

$\lambda = 0$, then (i) $\Rightarrow 2x - y + z - 3 = 0$

$\lambda = \frac{-24}{5}$, then (i) $\Rightarrow 5(2x - y + z - 3) - 24(3x + y + z - 5) = 0$

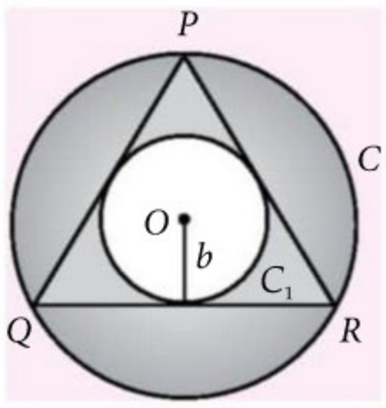
$\Rightarrow\ 62x + 29y + 19z - 105 = 0.$

11. (a, b, c) : All the sides of ΔPQR touch C_1.

$\therefore$ C_1 is incircle and C is the circumcircle with same centre.

The triangle is equilateral.

$R = 2r \Rightarrow a = 2b$



Area of $\Delta PQR = \frac{\sqrt{3}}{4}PR^2$

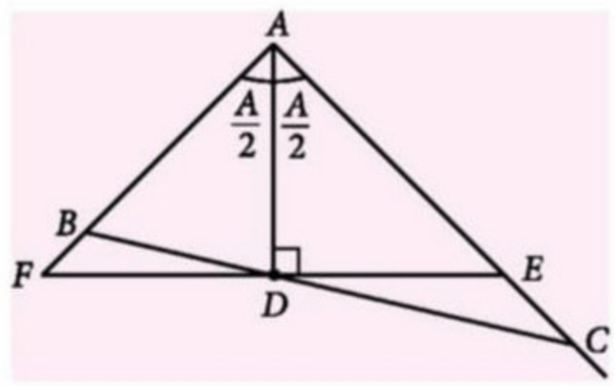
But $3b = \frac{\sqrt{3}}{2}PR$

$\therefore$ Area of $\Delta PQR = \frac{\sqrt{3}}{4} \cdot 4 \cdot 3b^2 = 3\sqrt{3}b^2$

$= \frac{3}{2}\sqrt{3}ab = \frac{3\sqrt{3}}{4}a^2$

12. (a, b, c, d) : $\Delta ABC = \Delta ABD + \Delta ADC$

$\therefore\ \frac{1}{2} \cdot bc\sin A = \frac{1}{2}c \cdot AD\sin\frac{A}{2} + \frac{1}{2}b \cdot AD\sin\frac{A}{2}$



$\Rightarrow\ bc \cdot 2\cos\frac{A}{2} = (b + c)AD$

$\therefore\ AD = \frac{2bc}{b + c}\cos\frac{A}{2}$ ...(1)

$AE\cos\frac{A}{2} = AD \Rightarrow AE = \frac{2bc}{b + c}$

$\therefore$ AE is H.M. of b and c ...(2)

$EF = 2ED = 2AD\tan\frac{A}{2} = \frac{4bc}{b + c}\sin\frac{A}{2}$ (using (1))

$AD \perp EF$, AD is angle bisector of $\angle A$

$\therefore$ AEF is isosceles.

13. (b, c) : a, b, c are in A.P. $\Rightarrow a + c = 2b$ ...(1)

b, c, d are in G.P. $\Rightarrow bd = c^2$ ...(2)

c, d, e are in H.P. $\Rightarrow d = \dfrac{2ce}{c+e}$ ...(3)

Since $a = 2$, $e = 18$, $b = \dfrac{2+c}{2}$, $d = \dfrac{36c}{c+18}$.

From (2), $\dfrac{18c(2+c)}{c+18} = c^2 \Rightarrow c^2 = 36$

$\Rightarrow \quad c = \pm 6 \Rightarrow b = \dfrac{2+6}{2}, \dfrac{2-6}{2} = 4, -2.$

14. (6) : $3^{2n+2} - 8n - 9 = 9^{n+1} - 8(n+1) - 1$

$= (8+1)^{n+1} - 8(n+1) - 1$

$= 8^{n+1} + \binom{n+1}{1}8^n + \ldots + \binom{n+1}{n-1}8^2$, which is divisible by $8^2 = 2^6$.

15. (2) : $e^3 = \exp \lim\limits_{x \to 0} \left(\dfrac{x + \dfrac{f(x)}{x}}{x} \right)$

$\Rightarrow \quad 3 = \lim\limits_{x \to 0} 1 + \dfrac{f(x)}{x^2} \Rightarrow \lim\limits_{x \to 0} \dfrac{f(x)}{x^2} = 2$

16. (0) : $x = 0$ in the identity gives

$$t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = -12 + 12 = 0.$$

17. (3) : $I_n = \int\limits_0^1 (x-1)^n e^x dx$

$= (x-1)^n e^x \Big|_0^1 - n \int\limits_0^1 (x-1)^{n-1} e^x \, dx$

$\Rightarrow \quad I_n = (-1)^{n+1} - nI_{n-1} \Rightarrow I_0 = \int\limits_0^1 e^x dx = e - 1$

$I_1 = 1 - I_0 = 1 - (e-1) = 2 - e$

$I_2 = -1 - 2I_1 = -1 - 2(2-e) = 2e - 5$

$I_3 = 1 - 3I_2 = 1 - 3(2e-5) = 16 - 6e \quad \therefore \; n = 3.$

18. (2) : ω, ω^2 are the roots of the equation

$$\frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} = \frac{2}{x} \qquad \ldots(i)$$

$\Rightarrow \quad x \Sigma (b+x)(c+x) = 2(a+x)(b+x)(c+x)$

$\Rightarrow \quad x^3 - (ab + bc + ca)x - 2abc = 0$

Coeff. of $x^2 = 0 \Rightarrow$ sum of the roots $= 0$

Also, $1 + \omega + \omega^2 = 0$

$\therefore$ The third root is 1.

(i) $\Rightarrow \quad \dfrac{1}{a+1} + \dfrac{1}{b+1} + \dfrac{1}{c+1} = \dfrac{2}{1} = 2.$

PAPER-2

1. (d) : Let the plane be $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

Volume $= k = \dfrac{abc}{6} \Rightarrow abc = 6k$ ....(i)

The foot of perpendicular from O on it is

$$\frac{x}{1/a} = \frac{y}{1/b} = \frac{z}{1/c} = \frac{1}{\left(\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}\right)} = \alpha, \text{ say}$$

$\therefore \quad x^2 + y^2 + z^2 = \alpha$

$\Rightarrow \quad ax = by = cz = x^2 + y^2 + z^2$ ...(ii)

From (i) and (ii), $6k = \dfrac{(x^2 + y^2 + z^2)^3}{xyz}$

or $(x^2 + y^2 + z^2)^3 = 6\, kxyz.$

2. (c): Let x be the length of an edge of the base and y be the height.

Given, $x^2 + 4xy = 400$ ...(i)

Volume, $V = x^2 y \Rightarrow V = \dfrac{x}{4}(400 - x^2)$

Now, $\dfrac{dV}{dx} = 0$

$\Rightarrow \quad \dfrac{1}{4}(400 - x^2) + \dfrac{x}{4}(-2x) = 0$

$\Rightarrow \quad \dfrac{-x^2}{4} + 100 - \dfrac{x^2}{2} = 0$

$\Rightarrow \quad x = \dfrac{20}{\sqrt{3}}, y = \dfrac{10}{\sqrt{3}} \quad \therefore \quad \dfrac{x}{y} = 2$

3. (c): $S = \dfrac{1}{2} \sum\limits_{r=1}^{n-1} \left(1 + \cos \dfrac{2r\pi}{n}\right)$

$= \dfrac{1}{2}\left[n - 1 + \sum\limits_{r=1}^{n-1} \cos \dfrac{2r\pi}{n}\right]$

Let $T = \sum\limits_{r=1}^{n-1} \cos \dfrac{2r\pi}{n}$

$\Rightarrow \quad 2 \sin \dfrac{\pi}{n} \cdot T = \sum\limits_{r=1}^{n-1} 2 \sin \dfrac{\pi}{n} \cos \dfrac{2r\pi}{n}$

$= \sum\limits_{r=1}^{n-1} \left(\sin(2r+1)\dfrac{\pi}{n} - \sin(2r-1)\dfrac{\pi}{n} \right)$

$= \sin(2n-1)\dfrac{\pi}{n} - \sin \dfrac{\pi}{n} = -2 \sin \dfrac{\pi}{n}$

$\therefore \quad T = -1, S = \dfrac{1}{2}(n - 1 - 1) = \dfrac{n-2}{2}.$

4. (d) : Let $a, ar, ar^2, \ldots\ldots$ be the G.P. $S = \dfrac{a(1-r^n)}{1-r}$

The reciprocals are in G. P. *i.e.* $\dfrac{1}{a}, \dfrac{1}{ar}, \dfrac{1}{ar^2}, \ldots..$

$$R = \frac{\frac{1}{a}\left(1-\frac{1}{r^n}\right)}{1-\frac{1}{r}} = \frac{1}{a}\left(\frac{1-r^n}{1-r}\right)\frac{1}{r^{n-1}}$$

$$\therefore \quad \frac{S}{R} = a^2 r^{n-1} \qquad \ldots.(1)$$

$$P = a \cdot ar \cdot ar^2 \ldots ar^{n-1} = a^n r^{(1+2+\ldots+(n-1))}$$
$$= a^n\, r^{(n-1)n/2}$$

$$\therefore \quad P^2 = a^{2n} \cdot r^{n(n-1)} = (a^2 r^{n-1})^n = \left(\frac{S}{R}\right)^n \quad \text{(by (1))}$$

$$\Rightarrow \quad S = R \cdot P^{2/n}$$

5. (c): $f'(x) = f(x) \Rightarrow f(x) = Ae^x$

$f(0) = 1 \Rightarrow f(x) = e^x$, $g(x) = x^2 - e^x$

$$I = \int_0^1 e^x (x^2 - e^x)dx = \int_0^1 (x^2 e^x - e^{2x})dx$$

$$= (x^2 - 2x + 2)e^x - \frac{e^{2x}}{2}\Big|_0^1$$

$$= e - 2 - \frac{e^2}{2} + \frac{1}{2} = e - \frac{e^2}{2} - \frac{3}{2}.$$

6. (b) : $\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0 \Rightarrow k = \dfrac{33}{2}$

By cross multiplication rule using the last 2 equations,

$$\frac{x}{6-4k} = \frac{y}{8} = \frac{z}{9-2k}$$

$$\Rightarrow \quad \frac{x}{15} = \frac{y}{-2} = \frac{z}{6} \Rightarrow \frac{xy}{z^2} = \frac{-30}{36} = \frac{-5}{6}.$$

7. (a, b, d) : For $y \geq 0$, $x + y = 2y \Rightarrow y = x$

For $y < 0$, $x - y = 2y \Rightarrow y = \dfrac{x}{3}$

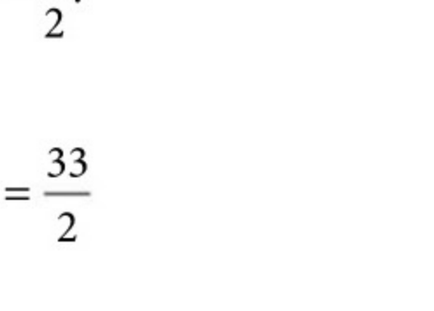
$f(x)$ is defined for all x.

$f(x)$ is continuous for all x.

$f(x)$ is differentiable for all x except $x = 0$.

$f'(x) = 1,\ x > 0,$

$f'(x) = \dfrac{1}{3},\ x < 0$



8. (a, c): $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = \cos\theta$

$$\Rightarrow \quad \vec{g} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}, \ |\vec{g}| = \frac{1}{3}\sqrt{|\vec{a}+\vec{b}+\vec{c}|^2}$$

$$= \frac{1}{3}\sqrt{1+1+1+6\cos\theta} = \frac{1}{\sqrt{3}}\sqrt{1+2\cos\theta} \qquad \ldots.(i)$$

$$\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \Rightarrow 0 \leq \cos\theta \leq \frac{1}{2},$$

$0 \leq 2\cos\theta \leq 1,\ 1 \leq 1 + 2\cos\theta \leq 2$

$$\therefore \ |\vec{g}| \in \left[\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right]$$

9. (a, c, d) : C divides AB in the ratio $t : 1 - t$

$$\therefore \quad |z - z_1| = t\,|z_1 - z_2| \Rightarrow |z - z_2| = (1-t)\,|z_1 - z_2|$$

Adding, $|z - z_1| + |z - z_2| = |z_1 - z_2|$

AC and AB lie along the same line.

$\therefore$ $\arg(z - z_1) = \arg(z_2 - z_1) = \theta$, say

$$z - z_1 = t(z_2 - z_1) = t|z_2 - z_1|\, e^{i\theta} = AB \cdot t \cdot e^{i\theta}$$

$$\begin{vmatrix} z - z_1 & \overline{z} - \overline{z}_1 \\ z_2 - z_1 & \overline{z}_2 - \overline{z}_1 \end{vmatrix} = \begin{vmatrix} AB \cdot t\, e^{i\theta} & ABt\, e^{-i\theta} \\ AB\, e^{i\theta} & AB\, e^{-i\theta} \end{vmatrix}$$

$$= (AB)^2 t \begin{vmatrix} e^{i\theta} & e^{-i\theta} \\ e^{i\theta} & e^{-i\theta} \end{vmatrix} = 0.$$

10. (a, c) : The foci of the ellipse are

$$\left(\pm\sqrt{a^2 - b^2}, 0\right) = (\pm 3, 0)$$

The eccentricity of the ellipse is

$$\sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

The eccentricity of hyperbola is $\dfrac{5}{3}$

Let the hyperbola be $\dfrac{x^2}{p^2} - \dfrac{y^2}{q^2} = 1$

It passes through $(\pm 3, 0)$

$\therefore \quad p^2 = 9$

$$q^2 = p^2(e^2 - 1) = 9\left(\frac{25}{9} - 1\right) = 16$$

$\therefore$ The hyperbola is $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

Foci are $\left(\pm\sqrt{p^2+q^2}, 0\right) = (\pm 5, 0)$.

11. (a, d) : $S_n < \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{n} \frac{1}{1+\frac{k}{n}+\left(\frac{k}{n}\right)^2}$

$$= \int_0^1 \frac{dx}{1+x+x^2} = \int_0^1 \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\Bigg|_0^1 = \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

$f(x)$ decreases in $(0, 1)$

$$\Rightarrow \quad \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) > \int_0^1 f(x)\,dx > \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

$$\therefore \quad T_n > \int_0^1 \frac{dx}{1+x+x^2} = \frac{\pi}{3\sqrt{3}}.$$

12. (b, c, d) : The tangent is $y - 1 = (2 + b)(x - 1)$

Its intercepts on the x and y-axis are $\frac{1+b}{2+b}$ and $-(1 + b)$

Now, $\frac{1}{2}\left|\frac{1+b}{2+b}\right||1+b| = 2$

$\Rightarrow \quad b^2 + 2b + 1 = \pm 4\,(2 + b)$

$\Rightarrow \quad (b + 3)^2 = 0,\ b^2 - 2b + 1 = 8 \Rightarrow b = -3, 1 \pm 2\sqrt{2}$

13. (b, d) : $x + y = 1$ ...(1)

$(c + 2)\,x + (c + 4)\,y = 6$ ...(2)

$(c + 2)^2\,x + (c + 4)^2\,y = 36$ ...(3)

(2) – (c + 2) (1) → (2), (3) – $(c + 2)^2$ (1) → (3) gives

$x + y = 1$ ...(1)

$2y = 4 - c$ ...(2)

$4(c + 3)\,y = (c + 8)\,(4 - c)$ ...(3)

(3) – 2 (c + 3) (2) → (3) gives

$0 = [(c + 8) - 2\,(c + 3)]\,(4 - c)$

$\Rightarrow \quad (4 - c)\,(2 - c) = 0 \Rightarrow c = 2, 4.$

14. (a, b, c) : Given $\frac{d}{dx}\left(y\frac{dy}{dx}\right) = x \Rightarrow y\frac{dy}{dx} = \frac{x^2}{2} + c$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{y^2}{2}\right) = \frac{x^2}{2} + c$$

$$\Rightarrow \quad y^2 = \frac{x^3}{3} + \alpha x + \beta \text{ given } y(0) = 1, y(1) = 1$$

$$\Rightarrow \quad \beta = 1, \alpha = \frac{-1}{3}$$

$$\therefore \quad y^2 = f(x) = \frac{x^3 - x + 3}{3},\ f'(x) = \frac{(3x^2 - 1)}{3} > 0 \text{ for } x > 1$$

$\therefore$ $f(x)$ is monotonically increasing $\forall\, x \in (1, \infty)$

$$f\left(\frac{1}{\sqrt{3}}\right) f\left(-\frac{1}{\sqrt{3}}\right) > 0$$

$f(x) = 0$ has only one real root.

15. (d) : Clearly $S_n > C_n > T_n$ as 'x' is a proper fraction.

$\therefore \quad x > x^2 > x^3$ and so on.

16. (c): We have $S_n = C_n + T_n$

$$\Rightarrow \quad x\frac{((x^3)^{2n} - 1)}{(x^3 - 1)} = x^2\frac{((x^3)^{2n} - 1)}{(x^3 - 1)} + x^3\frac{((x^3)^{2n} - 1)}{(x^3 - 1)}$$

But $x \neq 1$, as $x \in \left(0, \frac{\pi}{4}\right)$, so, we get $x = x^2 + x^3$

$\Rightarrow \quad x^2 + x - 1 = 0 \ (x \neq 0)$

$$\Rightarrow x = \frac{\sqrt{5}-1}{2} = 2\sin\frac{\pi}{10} \in \left(0, \frac{\pi}{4}\right)$$

17. (b) : Equation of hyperbola $(x - 1)(y - 2) = 4$

$\Rightarrow \quad XY = 4$

If normal at $\left(ct, \frac{c}{t}\right)$ intersect curve again at $\left(ct', \frac{c}{t'}\right)$

then $t' = -\frac{1}{t^3}$

$2t = 4 \quad \Rightarrow \quad t = 2$

$$(X', Y') \equiv \left(-\frac{1}{4}, -16\right)$$

$$\Rightarrow \quad (\alpha,\ \beta - 29) \equiv \left(\frac{3}{4}, -14\right) \Rightarrow \alpha = \frac{3}{4}, \beta = 15$$

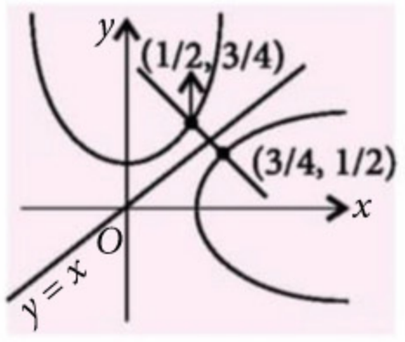
$$\text{so } \frac{\beta}{\alpha} = \frac{15\times 4}{3} = 20.$$

18. (a) : $2y\frac{dy}{dx} = 1 \ \Rightarrow \ \frac{dy}{dx} = \frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2}$

So, $d = \sqrt{\frac{1}{16} + \frac{1}{16}} = \frac{1}{2\sqrt{2}}$

So $|\sin\alpha| = 1$

So number of solutions is 3.



CONCEPT BOOSTERS

Trigonometry

This column is aimed for students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR

ELEMENTS OF A TRIANGLE

In a triangle ABC, the angles are denoted by capital letters A, B and C and the length of the sides opposite to these angles are denoted by small letters a, b and c. Semi perimeter of the triangle is given by $s=\frac{a+b+c}{2}$ and its area is denoted by Δ.

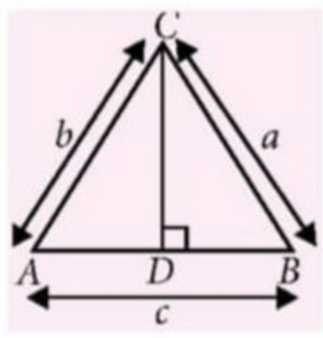
- $A + B + C = 180°$
- $\Delta=\frac{1}{2}$(base)(height)

SINE RULE AND AREA OF TRIANGLE

- In ΔBCD and ΔACD

  $CD = a \sin B = b \sin A$

  $\Rightarrow \quad \frac{a}{\sin A}=\frac{b}{\sin B}$



  Similarly $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

  Hence in ΔABC,

  $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$ (Sine rule)

  (where R is circumradius)

- Area of ΔABC,

  $\Delta=\frac{1}{2}\times AB\times CD=\frac{1}{2}\times c\times a\sin B=\frac{1}{2}ac\sin B$

  Similarly $\Delta=\frac{1}{2}bc\sin A=\frac{1}{2}ac\sin B=\frac{1}{2}ab\sin C.$

COSINE RULE

In ΔDBC

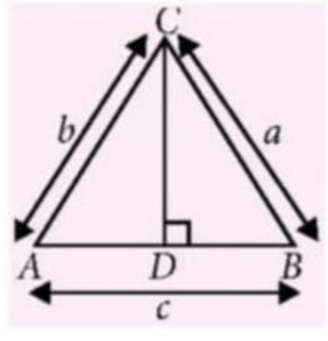
$BC^2 = DC^2 + DB^2$

$= DC^2 + (AB - AD)^2$

$= CD^2 + AD^2 + AB^2 - 2AB\cdot AD$

$= AC^2 + AB^2 - 2AB\cdot AC\cdot\cos A$

$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos A$



- $\cos A=\frac{b^2+c^2-a^2}{2bc}$
- $\cos B=\frac{c^2+a^2-b^2}{2ca}$
- $\cos C=\frac{a^2+b^2-c^2}{2ab}$

If $\angle A = 60°$, then $b^2 + c^2 - a^2 = bc$

If $\angle B = 60°$, then $a^2 + c^2 - b^2 = ac$

If $\angle C = 60°$, then $a^2 + b^2 - c^2 = ab$

PROJECTION FORMULAE

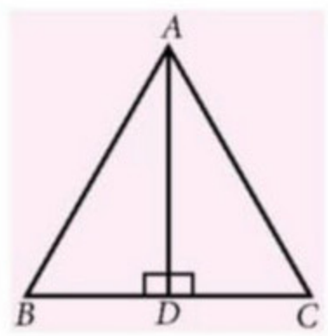
In ΔABC

$BC = BD + DC$

$= AB\cos B + AC\cos C$

$\Rightarrow \quad a = c\cos B + b\cos C$

- $a = b\cos C + c\cos B$
- $b = c\cos A + a\cos C$
- $c = a\cos B + b\cos A$



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NAPIER'S ANALOGY (TANGENT RULE)

From Sine formula,

$$\frac{b}{c}=\frac{\sin B}{\sin C}\Rightarrow\frac{b-c}{b+c}=\frac{\sin B-\sin C}{\sin B+\sin C}=\tan\frac{A}{2}.\tan\frac{B-C}{2}$$

$$\therefore\ \tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$$

- $\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$
- $\tan\left(\frac{C-A}{2}\right)=\frac{c-a}{c+a}\cot\frac{B}{2}$
- $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$

HALF ANGLE FORMULAE

$$\text{Since } 2\sin^2\frac{A}{2}=1-\cos A=1-\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}=\frac{(a+b-c)(a-b+c)}{2bc}$$

$$=\frac{(2s-2c)(2s-2b)}{2bc}$$

$$\therefore\ \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

- $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$
- $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$
- $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

$$\text{Since } 2\cos^2\frac{A}{2}=1+\cos A$$

$$=1+\frac{b^2+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(b+c+a)(b+c-a)}{2bc}=\frac{2s(2s-2a)}{2bc}$$

$$\therefore\ \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$$

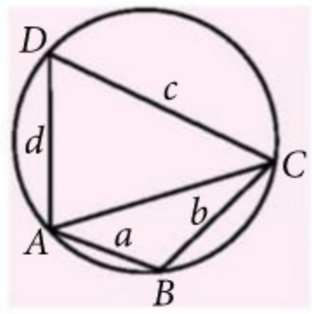
- $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$
- $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$
- $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

Using $\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$, we get

- $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$
- $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$
- $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

AREA OF CYCLIC QUADRILATERAL

Let $ABCD$ be a cyclic quadrilateral whose sides AB, BC, CD and DA are respectively a, b, c and d.



Area of quadrilateral

$$=\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Circumradius of a cyclic quadrilateral

$$R=\frac{1}{4\Delta}\sqrt{(ac+bd)(ad+bc)(ab+cd)}$$

CENTROID AND MEDIANS OF A TRIANGLE

The line joining any vertex of a triangle to the mid point of the opposite side of the triangle is called median of the triangle. The three medians of a triangle are concurrent and the point of concurrency of the medians of any triangle is called the centroid of the triangle. The centroid divides the median in the ratio 2 : 1.

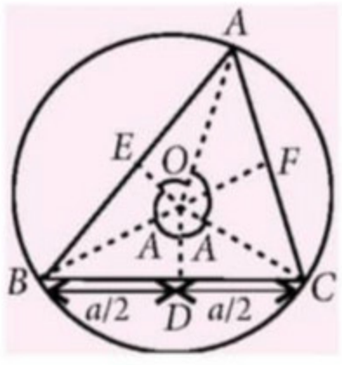
The distance of centroid from vertex A of triangle ABC, $GA=\frac{1}{3}\sqrt{2b^2+2c^2-a^2}$ and the distance of centroid from side BC of a triangle ABC, $G_a=\frac{2\Delta}{3a}$.

APOLLONIUS THEOREM :

In a triangle ABC, if AD is median through A, then $AB^2 + AC^2 = 2(AD^2 + BD^2)$

CIRCUMCIRCLE

The circle which passes through the angular points of a ΔABC, is called its circumcircle. The centre of this circle *i.e.*, the point of concurrency of the



perpendicular bisectors of the sides of the ΔABC, is called the circumcentre.

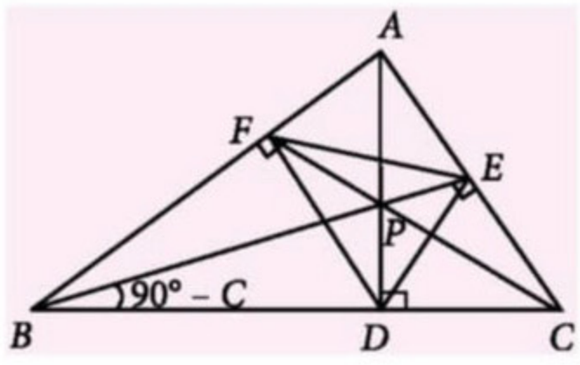
Radius of the circumcircle is given by the following formulae

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta}$$

Distance of circumcentre from vertex A of triangle $ABC = R$ and distance of circumcentre from side BC of triangle ABC is $R\cos A$.

ORTHOCENTRE AND PEDAL TRIANGLE OF A TRIANGLE.

In a triangle, the altitudes drawn from the three vertices to the opposite sides are concurrent and the point of cuncurrency of the altitudes of the triangle is called the orthocentre of the triangle.



$PD = 2R\cos B\cos C$ $\quad PE = 2R\cos A\cos C$

$PF = 2R\cos A\cos B$ $\quad PA = 2R\cos A$

$PB = 2R\cos B$ $\quad PC = 2R\cos C$

$$AD=\frac{1}{2}\sqrt{b^2+c^2+2bc\cos A}=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$$

$$BE=\frac{1}{2}\sqrt{c^2+a^2+2ac\cos B}=\frac{1}{2}\sqrt{2c^2+2a^2-b^2}$$

$$CF=\frac{1}{2}\sqrt{a^2+b^2+2ab\cos C}=\frac{1}{2}\sqrt{2a^2+2b^2-c^2}$$

The triangle formed by joining the feet of these perpendiculars is called the pedal triangle *i.e.*, ΔDEF is the pedal triangle of ΔABC.

Pedal triangle constructed on ΔABC w.r.t. orthocentre

- Its sides : $R\sin 2A$, $R\sin 2B$, $R\sin 2C$
- Its angles : $P-2A$, $P-2B$, $P-2C$
- Its circumradius = $R/2$ and inradius = $2R\cos A\cos B\cos C$

BISECTORS OF THE ANGLES

If AD bisects the angle A and divide the base into portions x and y, we have, by Geometry,

$$\frac{x}{y}=\frac{AB}{AC}=\frac{c}{b}\ \therefore\ \frac{x}{c}=\frac{y}{b}=\frac{x+y}{b+c}=\frac{a}{b+c}$$

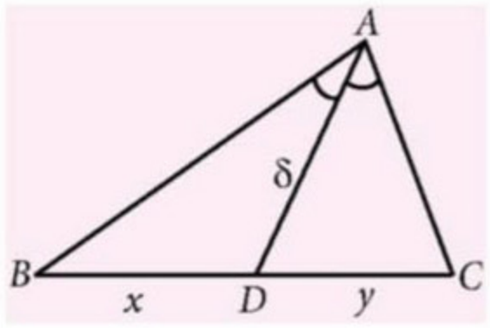
$$\Rightarrow\ x=\frac{ac}{b+c}\text{ and }y=\frac{ab}{b+c}$$

Also, let δ be the length of AD

we have Area of ΔABD + Area of ΔACD = Area of ΔABC

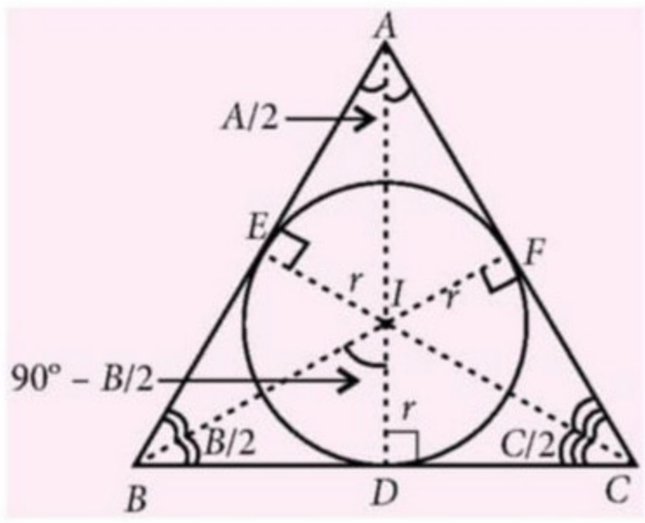
$$\therefore\ \frac{1}{2}c\delta\sin\frac{A}{2}+\frac{1}{2}b\delta\sin\frac{A}{2}=\frac{1}{2}bc\sin A,$$

$$i.e.,\ \delta=\frac{bc}{b+c}\frac{\sin A}{\sin\frac{A}{2}}=\frac{2bc}{b+c}\cos\frac{A}{2}$$



INCIRCLE

The circle which can be inscribed within the triangle so as to touch each of the sides of the triangle is called its incircle. The centre of this circle *i.e.*, the point of concurrency of angle bisectors of the triangle is called the incentre of the ΔABC.



Radius of the incircle is given by the following formulae :

$$r=\frac{\Delta}{s}=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}$$

$$=(s-c)\tan\frac{C}{2}=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$r=\frac{a\sin(B/2)\sin(C/2)}{\cos A/2}=\frac{b\sin(A/2)\sin(C/2)}{\cos B/2}$$

$$=\frac{c\sin(B/2)\sin(A/2)}{\cos(C/2)}$$

INSCRIBED & CIRCUMSCRIBED POLYGONS (IMPORTANT FORMULAE)

- Area of Polygon of n sides inscribed in a circle of radius $R=\frac{1}{2}nR^2\sin\frac{2\pi}{n}$
- Area of Polygon of n sides inscribing a circle of radius $r=\frac{1}{2}nr^2\tan\frac{\pi}{n}$

- Side of Inscribed polygon $= 2R\sin\frac{\pi}{n}$
- Side of Circumscribed polygon $= 2r\tan\frac{\pi}{n}$

INVERSE TRIGONOMETRIC FUNCTION

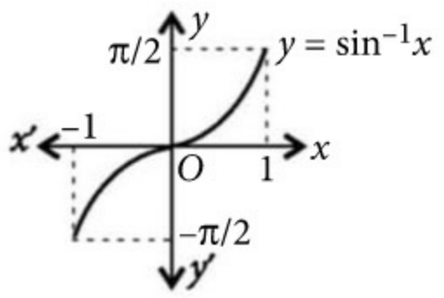
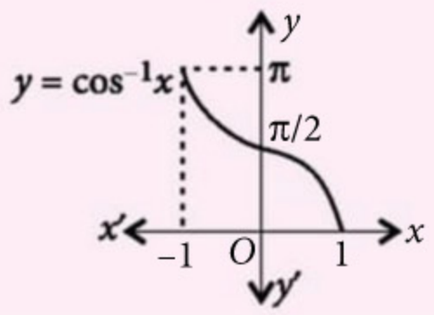
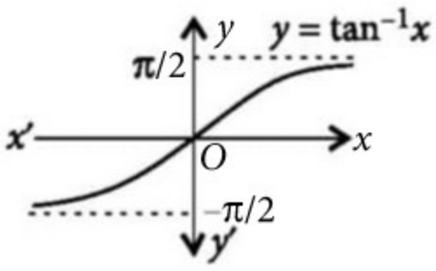
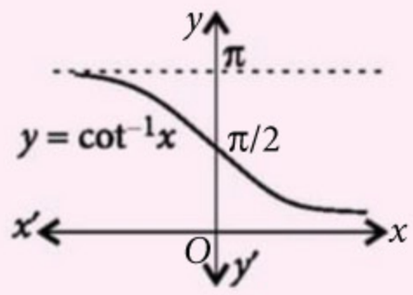
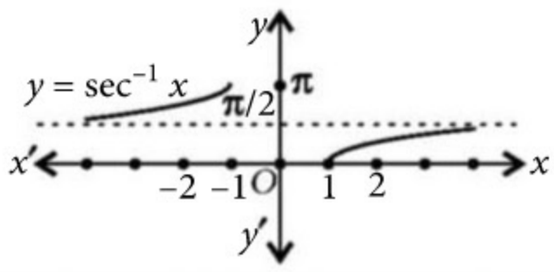
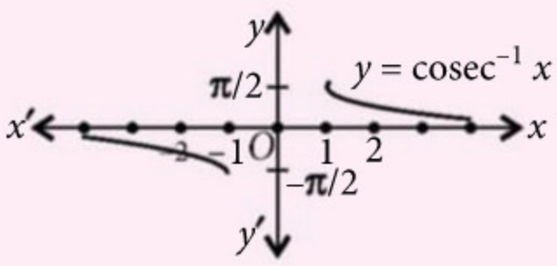
If $f : A \to B$ is one to one and onto function and g is a rule under which for every element $y \in B$, $\exists$ a unique element $x \in A$ then $g : B \to A$ is called Inverse function of $f : A \to B$.

i.e., $g = f^{-1}$

$\therefore \quad x = g(y) \implies x = f^{-1}(y)$

So $y = f(x)$ and $x = g(y)$ such that $f(g(y)) = y$ and $x = g(f(x))$ then f and g are said to be inverse function of each other.

DOMAIN, RANGE AND GRAPH OF INVERSE TRIGNOMETRIC FUNCTIONS

Function	Domain	Range	Graph
$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\pi/2 \le y \le \pi/2$	
$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$	
$y = \tan^{-1}x$	$x \in R$	$-\pi/2 < y < \pi/2$	
$y = \cot^{-1}x$	$x \in R$	$0 < y < \pi$	
$y = \sec^{-1}x$	$\lvert x\rvert \ge 1$	$0 \le y \le \pi,\ y \ne \pi/2$	
$y = \text{cosec}^{-1}x$	$\lvert x\rvert \ge 1$	$-\pi/2 \le y \le \pi/2,\ y \ne 0$	

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

Property I

- $\sin^{-1}(-x) = -\sin^{-1}x \ \forall\ x \in [-1, 1]$
- $\tan^{-1}(-x) = -\tan^{-1}x \ \forall\ x \in R$
- $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x \ \ \forall\ x \in (-\infty, -1] \cup [1, \infty)$
- $\cos^{-1}(-x) = \pi - \cos^{-1}x \ \forall\ x \in [-1, 1]$
- $\cot^{-1}(-x) = \pi - \cot^{-1}x \ \forall\ x \in R$
- $\sec^{-1}(-x) = \pi - \sec^{-1}x \ \ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

Property II

- $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x \ \forall\ x \in (-\infty, -1] \cup [1, \infty)$
- $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x \ \forall\ x \in (-\infty, -1] \cup [1, \infty)$
- $\tan^{-1}(1/x) = \begin{cases} \cot^{-1}x, x > 0 \\ -\pi + \cot^{-1}x, \ x < 0 \end{cases}$

Property III

- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \forall x \in [-1, 1]$
- $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}, \forall x \in (-\infty, -1] \cup [1, \infty)$
- $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \forall x \in R$

Property IV

- $\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\frac{x+y}{1-xy}, x > 0, y > 0, \\ \pi + \tan^{-1}\frac{x+y}{1-xy}, x > 0, \\ \qquad y > 0, xy > 1 \\ -\pi + \tan^{-1}\frac{x+y}{1-xy}, \ x < 0, \\ \qquad y < 0, xy > 1 \end{cases}$

- $\tan^{-1}x - \tan^{-1}y = \begin{cases} \tan^{-1}\frac{x-y}{1+xy}, xy > -1 \\ \pi + \tan^{-1}\frac{x-y}{1+xy}, x > 0, \\ \qquad y < 0 \& xy < -1 \\ -\pi + \tan^{-1}\frac{x-y}{1+xy}, \ x < 0, \\ \qquad y > 0 \& xy < -1 \end{cases}$

Remark : If $x_1, x_2, x_3, \ldots\ldots x_n \in R$, then

$$\tan^{-1}x_1 + \tan^{-1}x_2 + \ldots\ldots + \tan^{-1}x_n = \tan^{-1}\left(\frac{s_1 - s_3 + s_5 \ldots}{1 - s_2 + s_4 - s_6 \ldots}\right)$$

Where $s_1 = x_1 + x_2 + \ldots\ldots + x_n = \sum x_1$

$s_2 = x_1x_2 + x_2x_3 + \ldots\ldots + x_{n-1}x_n = \sum x_1x_2$

$s_3 = \sum x_1x_2x_3$ and so on.

Property V

- $\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}, \\ \text{if } x, y \geq 0 \ \& x^2 + y^2 \leq 1 \\ \text{or if } xy < 0 \ \& \ x^2 + y^2 > 1 \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} \right. \\ \qquad \left. + y\sqrt{1-x^2}\right\} \\ \text{if } 0 < x, y \leq 1 \ \& \ x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} \right. \\ \qquad \left. + y\sqrt{1-x^2}\right\} \\ \text{if } -1 \leq x, y < 0 \ \& \ x^2 + y^2 > 1 \end{cases}$

- $\sin^{-1}x - \sin^{-1}y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, \\ \text{if } x, y \geq 0 \ \& \ x^2 + y^2 \leq 1 \\ \text{or if } xy > 0 \ \& \ x^2 + y^2 > 1 \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} \right. \\ \qquad \left. - y\sqrt{1-x^2}\right\} \\ \text{if } 0 < x \leq 1, -1 \leq y < 0 \\ \qquad \& \ x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} \right. \\ \qquad \left. - y\sqrt{1-x^2}\right\} \\ \text{if } -1 \leq x < 0, 0 < y \leq 1 \\ \qquad \& \ x^2 + y^2 > 1 \end{cases}$

Property VI

- $\cos^{-1}x+\cos^{-1}y=\begin{cases}\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)\\ \text{if } x,y\geq 0 \ \&\ x+y\geq 0\\ 2\pi-\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)\\ \text{if } -1\leq x,y\leq 1 \& x+y\leq 0\end{cases}$

- $\cos^{-1}x-\cos^{-1}y=\begin{cases}\cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right)\\ \text{if } x,y\geq 0,\ x\leq y\\ -\cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right)\\ \text{if } 0<x\leq 1,-1\leq y\leq 0\ \&\ x\geq y\end{cases}$

Property VII

- $\sin^{-1}(2x\sqrt{1-x^2})\begin{cases}2\sin^{-1}x & \text{if } \frac{-1}{\sqrt{2}}\leq x\leq\frac{1}{\sqrt{2}}\\ \pi-2\sin^{-1}x, & \text{if } \frac{1}{\sqrt{2}}\leq x\leq 1\\ -\pi-2\sin^{-1}x & \text{if } -1\leq x\leq\frac{-1}{\sqrt{2}}\end{cases}$

- $\cos^{-1}(2x^2-1)=\begin{cases}2\cos^{-1}x \text{ if } 0\leq x\leq 1\\ 2\pi-2\cos^{-1}x \text{ if } -1\leq x\leq 0\end{cases}$

- $\sin^{-1}\frac{2x}{1+x^2}=\begin{cases}2\tan^{-1}x \ \text{ if } -1\leq x\leq 1\\ \pi-2\tan^{-1}x \ \text{ if } x>1\\ -\pi-2\tan^{-1}x \text{ if } x<-1\end{cases}$

- $\cos^{-1}\frac{1-x^2}{1+x^2}=\begin{cases}2\tan^{-1}x \text{ if } x\geq 0\\ -2\tan^{-1}x \text{ if } x\leq 0\end{cases}$

- $\tan^{-1}\frac{2x}{1-x^2}=\begin{cases}2\tan^{-1}x, \text{ if } -1\leq x\leq 1\\ \pi-2\tan^{-1}x, \text{if } x<-1\\ -\pi+2\tan^{-1}x \ \text{ if } x>1\end{cases}$

Property VIII

- $\sin^{-1}(3x-4x^3)=\begin{cases}3\sin^{-1}x \ \ ;\frac{-1}{2}\leq x\leq\frac{1}{2}\\ \pi-3\sin^{-1}x; \ \frac{1}{2}\leq x\leq 1\\ -\pi-3\sin^{-1}x; -1\leq x\leq\frac{-1}{2}\end{cases}$

- $\cos^{-1}(4x^3-3x)=\begin{cases}-2\pi+3\cos^{-1}x; -1\leq x\leq\frac{-1}{2}\\ 2\pi-3\cos^{-1}x; \ \frac{-1}{2}\leq x\leq\frac{1}{2}\\ 3\cos^{-1}x \quad ; \quad \frac{1}{2}\leq x\leq 1\end{cases}$

- $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)=\begin{cases}\pi+3\tan^{-1}x, \ x<\frac{-1}{\sqrt{3}}\\ 3\tan^{-1}x, \ \frac{-1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}\\ -\pi+3\tan^{-1}x, \ x>\frac{1}{\sqrt{3}}\end{cases}$

Property IX : Conversion

- $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\text{cosec}^{-1}\left(\frac{1}{x}\right)$

- $\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=$

$\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)=\sec^{-1}\left(\frac{1}{x}\right)=\text{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

- $\tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$=\cot^{-1}\left(\frac{1}{x}\right)=\sec^{-1}\left(\sqrt{1+x^2}\right)=\text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

PROBLEMS

Single Option Correct Answer Type

1. If $x=\log\left[\cot\left(\frac{\pi}{4}+\theta\right)\right]$, then $\sinh x=$

(a) $\tan 2\theta$ (b) $\cot 2\theta$

(c) $-\tan 2\theta$ (d) $-\cot 2\theta$

2. AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point C on the ground is $60°$. He moves away from the pole along the line BC to a point D such that $CD = 7$ m. From D the angle of

elevation of the point A is 45°. Then the height of the pole is

(a) $\frac{7\sqrt{3}}{2}(\sqrt{3}+1)$m (b) $\frac{7\sqrt{3}}{2}(\sqrt{3}-1)$m

(c) $\frac{7\sqrt{3}}{2}\frac{1}{\sqrt{3}+1}$m (d) $\frac{7\sqrt{3}}{2}\frac{1}{\sqrt{3}-1}$m

3. The angle of elevation of a cloud from a point h m above the surface of a lake is θ and the angle of depression of its reflection in the lake is ϕ. The height of the cloud is

(a) $\frac{h\sin(\phi+\theta)}{\sin(\phi-\theta)}$ (b) $\frac{h\sin(\phi-\theta)}{\sin(\phi+\theta)}$

(c) $\frac{h\sin(\theta+\phi)}{\sin(\theta-\phi)}$ (d) $\frac{h\sin(\theta-\phi)}{\sin(\theta+\phi)}$

4. If $\tan\left(\frac{\pi}{4}+\frac{y}{2}\right)=\tan^3\left(\frac{\pi}{4}+\frac{x}{2}\right)$, then

$\sin x\left(\frac{3+\sin^2 x}{1+3\sin^2 x}\right)$ equals

(a) $\cos y$ (b) $\sin y$ (c) $\sin 2y$ (d) 0

5. If $x_1, x_2, x_3, ..., x_n$ are in A.P, whose common difference is α, then the value of

$\sin\alpha[\sec x_1 \sec x_2+\sec x_2 \sec x_3+.....+\sec x_{n-1}\sec x_n]=$

(a) $\frac{\sin n\alpha}{\cos x_n \cos x_1}$ (b) $\frac{\sin(n-1)\alpha}{\cos x_n \cos x_1}$

(c) $\frac{\sin(n+1)\alpha}{\cos x_n \cos x_1}$ (d) $\frac{\cos(n-1)\alpha}{\cos x_n \cos x_1}$

6. If $a\sin^2 x + b\cos^2 x = c$, $b\sin^2 y + a\cos^2 y = d$ and $a\tan x = b\tan y$, then $\frac{a^2}{b^2}=$

(a) $\frac{(a-d)(c-a)}{(b-c)(d-b)}$ (b) $\frac{(a+d)(c+a)}{(b+c)(d+b)}$

(c) $\frac{(a-d)(b-a)}{(a-c)(c-b)}$ (d) $\frac{(d-a)(c-a)}{(b-c)(d-b)}$

7. If $\cos^3 x\sin 2x=\sum_{r=0}^{n}a_r\sin(rx),\forall x\in R$ then

(a) $n=5, a_1=\frac{1}{2}$ (b) $n=5, a_1=\frac{1}{4}$

(c) $n=5, a_2=\frac{1}{8}$ (d) $n=5, a_2=\frac{1}{4}$

8. If $\cos\theta=\frac{a\cos\phi+b}{a+b\cos\phi}$, then $\tan\theta/2$ is equal to

(a) $\sqrt{\left(\frac{a-b}{a+b}\right)}\tan(\phi/2)$ (b) $\sqrt{\left(\frac{a+b}{a-b}\right)}\cos(\phi/2)$

(c) $\sqrt{\left(\frac{a-b}{a+b}\right)}\sin(\phi/2)$ (d) none of these

9. The value of $\sum_{r=0}^{10}\cos^3\frac{\pi r}{3}$ is equal to

(a) $\frac{-9}{2}$ (b) $\frac{-7}{2}$ (c) $\frac{-9}{8}$ (d) $\frac{-1}{8}$

10. The value of $\sin^3 10° + \sin^3 50° - \sin^3 70°$ is equal to

(a) $-\frac{3}{2}$ (b) $\frac{3}{4}$ (c) $-\frac{3}{4}$ (d) $-\frac{3}{8}$

11. If $\tan(\alpha-\beta)=\frac{\sin 2\beta}{3-\cos 2\beta}$, then

(a) $\tan\alpha = 2\tan\beta$ (b) $\tan\beta = 2\tan\alpha$

(c) $2\tan\alpha = 3\tan\beta$ (d) $3\tan\alpha = 2\tan\beta$

12. In a triangle ABC, if angle C is obtuse and angles A and B are given by roots of the equation $\tan^2 x + p\tan x + q = 0$, then the value of q is

(a) greater than 1 (b) less than 1

(c) equal to 1 (d) 0

13. If $2\sin x - \cos 2x = 1$, then $\cos^2 x + \cos^4 x$ is equal to

(a) 1 (b) – 1 (c) $-\sqrt{5}$ (d) $\sqrt{5}$

14. If $ABCD$ is a cyclic quadrilateral such that $13\cos A + 12 = 0$ and $3\tan B - 4 = 0$, then the quadratic equation whose roots are $\tan C$ and $\cos D$ is

(a) $15x^2 + 60x - 11 = 0$ (b) $60x^2 + 11x - 15 = 0$

(c) $11x^2 + 60x - 15 = 0$ (d) none of these

15. If A, B, C are the angles of a triangle such that $\cot\frac{A}{2}=3\tan\frac{C}{2}$, then $\sin A$, $\sin B$, $\sin C$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

16. If $\frac{2\tan\alpha}{1+\sec\alpha+\tan\alpha}=\lambda$, then $\frac{2\tan\alpha/2}{1+\tan\alpha/2}$ is equal to

(a) $\frac{1}{\lambda}$ (b) λ (c) $1-\lambda$ (d) $1+\lambda$

17. If $0 \le A, B, C \le \pi$ and $A + B + C = \pi$, then the minimum value of $\sin 3A + \sin 3B + \sin 3C$ is

(a) –2 (b) $-\frac{3\sqrt{3}}{2}$

(c) 0 (d) none of these

18. The value of x which satisfies the equation $2\tan^{-1}2x=\sin^{-1}\dfrac{4x}{1+4x^2}$ is

(a) $\left[\dfrac{1}{2},\infty\right)$ (b) $\left(-\infty,-\dfrac{1}{2}\right]$

(c) $[-1, 1]$ (d) $\left[-\dfrac{1}{2},\dfrac{1}{2}\right]$

19. Number of integral solutions of the equation $3\tan^{-1}x+\cos^{-1}\left(\dfrac{1-3x^2}{(1+x^2)^{3/2}}\right)=0$ is

(a) 1 (b) 2 (c) 0 (d) infinite

20. In a triangle ABC, with $A=\dfrac{\pi}{7},B=\dfrac{2\pi}{7},C=\dfrac{4\pi}{7}$, then $a^2+b^2+c^2$ is (R = circumradius of ΔABC)

(a) $4R^2$ (b) $6R^2$ (c) $7R^2$ (d) $8R^2$

21. For what value of x, $\sin(\cot^{-1}(x+1))=\cos(\tan^{-1}x)$?

(a) $\dfrac{1}{2}$ (b) 0 (c) 1 (d) $-\dfrac{1}{2}$

22. If the equation $x^2+12+3\sin(a+bx)+6x=0$ has atleast one real solution where $a, b\in[0, 2\pi]$, then value of $\cos\theta$, where θ is least positive value of $a+bx$ is

(a) π (b) 2π (c) 0 (d) $\dfrac{\pi}{2}$

23. In any ΔABC, which is not right angled, $\Sigma\cos A\ \mathrm{cosec}B\ \mathrm{cosec}C$ is

(a) constant (b) less than 1

(c) greater than 2 (d) none of these

24. If $\angle C=90^\circ$ in ΔABC, then

$\tan^{-1}\left(\dfrac{a}{b+c}\right)+\tan^{-1}\left(\dfrac{b}{c+a}\right)$ is equal to

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{3}$ (d) π

Multiple Option Correct Answer Type

25. In ΔPQR, if $3\sin P+4\cos Q=6$ and $4\sin Q+3\cos P=1$ then $\angle R$ can be

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{3\pi}{4}$ (c) $\dfrac{5\pi}{6}$ (d) $\dfrac{\pi}{4}$

26. If $\left(\dfrac{\sin\theta}{\sin\phi}\right)^2=\dfrac{\tan\theta}{\tan\phi}=3$, then

(a) $\tan\phi=\dfrac{1}{\sqrt{3}}$ (b) $\tan\phi=-\dfrac{1}{\sqrt{3}}$

(c) $\tan\theta=\sqrt{3}$ (d) $\tan\theta=-\sqrt{3}$

27. Which of the following quantities are rational?

(a) $\sin\left(\dfrac{11\pi}{12}\right)\sin\left(\dfrac{5\pi}{12}\right)$

(b) $\mathrm{cosec}\left(\dfrac{9\pi}{10}\right)\sec\left(\dfrac{4\pi}{5}\right)$

(c) $\sin^4\left(\dfrac{\pi}{8}\right)+\cos^4\left(\dfrac{\pi}{8}\right)$

(d) $\left(1+\cos\dfrac{2\pi}{9}\right)\left(1+\cos\dfrac{4\pi}{9}\right)\left(1+\cos\dfrac{8\pi}{9}\right)$

28. A solution (x, y) of the system of equations $x-y=\dfrac{1}{3}$ and $\cos^2(\pi x)-\sin^2(\pi y)=\dfrac{1}{2}$ is

(a) $\left(\dfrac{7}{6},\dfrac{5}{6}\right)$ (b) $\left(\dfrac{2}{3},\dfrac{1}{3}\right)$

(c) $\left(\dfrac{-5}{6},\dfrac{-7}{6}\right)$ (d) $\left(\dfrac{13}{6},\dfrac{11}{6}\right)$

29. If $0\le x\le 2\pi$ then $2^{\mathrm{cosec}^2x}\sqrt{\dfrac{y^2}{2}-y+1}\le\sqrt{2}$ is

(a) satisfied by exactly one value of y

(b) satisfied by exactly two values of x

(c) satisfied by x for which $\cos x = 0$

(d) satisfied by x for which $\sin x = 0$

30. If $x\cos\alpha+y\sin\alpha=x\cos\beta+y\sin\beta=2a$ and $2\sin\left(\dfrac{\alpha}{2}\right)\sin\left(\dfrac{\beta}{2}\right)=1$ then

(a) $y^2=4a(a-x)$

(b) $\cos\alpha+\cos\beta=\cos\alpha\cos\beta$

(c) $\cos\alpha\cdot\cos\beta=\dfrac{4a^2+y^2}{x^2+y^2}$

(d) $\cos\alpha+\cos\beta=\dfrac{4ax}{x^2+y^2}$

31. If $\dfrac{\tan 3A}{\tan A}=k\,(k\ne 1)$, then

(a) $\dfrac{\cos A}{\cos 3A}=\dfrac{k-1}{2}$ (b) $\dfrac{\sin 3A}{\sin A}=\dfrac{2k}{k-1}$

(c) $k<\dfrac{1}{3}$ (d) $k>3$

32. Let $f_n(\theta)=\tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)$

$(1+\sec 4\theta)....(1+\sec 2^n\theta)$, then

(a) $f_2\left(\frac{\pi}{16}\right)=1$ (b) $f_3\left(\frac{\pi}{32}\right)=1$

(c) $f_4\left(\frac{\pi}{64}\right)=1$ (d) $f_5\left(\frac{\pi}{128}\right)=1$

33. Sum of series

$$\sum_{r=1}^{n}\sin^{-1}\left[\frac{2r+1}{r(r+1)\left(\sqrt{r^2+2r}+\sqrt{r^2-1}\right)}\right]$$ is

(a) $\frac{\pi}{2}-\sin^{-1}\left(\frac{1}{n+1}\right)$ (b) $\cos^{-1}\left(\frac{1}{n+1}\right)$

(c) $\cos^{-1}\left(\frac{1}{n+2}\right)$ (d) none of these

34. Minimum positive values of x and y such that $x+y=\frac{\pi}{2}$ and $\sec x+\sec y=2\sqrt{2}$ are

(a) $x=\frac{\pi}{4}$ (b) $y=\frac{\pi}{4}$

(c) $x=-\frac{\pi}{4}$ (d) $y=-\frac{\pi}{4}$

Comprehension Type

Paragraph for Question No. 35 to 37

Let $\cos\frac{\pi}{7},\cos\frac{3\pi}{7},\cos\frac{5\pi}{7}$ are the roots of equation $8x^3-4x^2-4x+1=0$

35. The value of $\sec\left(\frac{\pi}{7}\right)+\sec\left(\frac{3\pi}{7}\right)+\sec\left(\frac{5\pi}{7}\right)$ is

(a) 2 (b) 4

(c) 8 (d) none of these

36. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{7}}{8}$

37. The value of $\cos\left(\frac{\pi}{14}\right)\cos\left(\frac{3\pi}{14}\right)\cos\left(\frac{5\pi}{14}\right)$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{8}$ (c) $\frac{\sqrt{7}}{2}$ (d) $\frac{\sqrt{7}}{8}$

Matrix-Match Type

38. Match the following :

Column I		Column II	
(A)	If $\tan\theta$ is the G.M. between $\sin\theta$ and $\cos\theta$, then $2-4\sin^2\theta+3\sin^4\theta-\sin^6\theta=$	(p)	1
(B)	$\sqrt{3}\cot 20^\circ-4\cos 20^\circ=$	(q)	0
(C)	$\cot 16^\circ\cot 44^\circ+\cot 44^\circ\cot 76^\circ-\cot 76^\circ\cot 16^\circ=$	(r)	3
(D)	$\sum_{r=1}^{9}\sin^2\left(\frac{r\pi}{18}\right)=$	(s)	5

39. Match the following trigonometric ratios with the equations whose one of the roots is given :

Column I		Column II	
(A)	$\cos 20^\circ$	(p)	$x^3-3x^2-3x+1=0$
(B)	$\sin 10^\circ$	(q)	$32x^5-40x^3+10x-1=0$
(C)	$\tan 15^\circ$	(r)	$8x^3-6x-1=0$
(D)	$\sin 6^\circ$	(s)	$8x^3-6x+1=0$

40. Match the following :

Column I		Column II	
(A)	If $\sin\theta=3\sin(\theta+2\alpha)$, then the value of $\tan(\theta+\alpha)+2\tan\alpha$ is	(p)	0
(B)	If $p\sin\theta+q\cos\theta=a$ and $p\cos\theta-q\sin\theta=b$, then $\frac{p+a}{q+b}+\frac{q-b}{p-a}+1$ is equal to	(q)	1
(C)	The value of the expression $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{10\pi}{7}-\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is	(r)	$\sec\theta$
(D)	If $\sec\theta+\tan\theta=1$, then one root of the equation $(a-2b+c)x^2+(b-2c+a)x+(c-2a+b)=0$ is	(s)	$-\frac{1}{4}$
		(t)	$-\frac{1}{2}$

Numerical Value Answer Type

41. Let $f(x) = 0$ be an equation of degree six, having integer coefficients and whose one root is $2\cos\frac{\pi}{18}$. Then, the sum of all the roots of $f'(x) = 0$, is

42. If $\cos\theta + \cos^2\theta + \cos^3\theta = 1$ and $\sin^6\theta = a + b\sin^2\theta + c\sin^4\theta$, then $a + b + c =$

43. If $\frac{\sin\theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta} = \frac{1}{A}[\tan B\theta - \tan C\theta]$, then $(27A - B - 27C) =$

44. If $\frac{\tan x}{2} = \frac{\tan y}{3} = \frac{\tan z}{5}$ and $x + y + z = \pi$, $\tan^2 x + \tan^2 y + \tan^2 z = \frac{38}{K}$, then $K =$

45. If $\alpha + \beta + \gamma = \pi$ and

$$\tan\left[\frac{\alpha+\beta-\gamma}{4}\right]\tan\left[\frac{\gamma+\alpha-\beta}{4}\right]\tan\left[\frac{\gamma+\beta-\alpha}{4}\right] = 1,$$

then the value of $1 + \cos\alpha + \cos\beta + \cos\gamma$ is $K - 1$, where K is

SOLUTIONS

1. (c) : $x = \log[\cot(\pi/4 + \theta)]$

$$= \log\left[\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right] \Rightarrow e^x = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\frac{(\cos\theta - \sin\theta)^2 - (\cos\theta + \sin\theta)^2}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}\right]$$

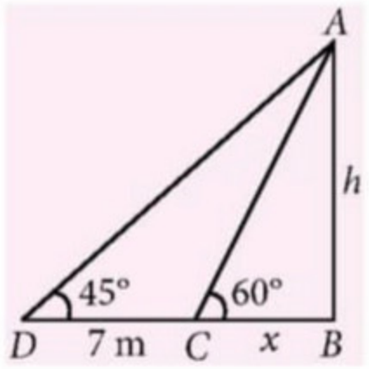
$$= \frac{1}{2}\left[\frac{-4\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta}\right] = \frac{-\sin 2\theta}{\cos 2\theta} = -\tan 2\theta$$

2. (a) : $x = h\cot 60° = h/\sqrt{3}$

$x + 7 = h\cot 45°$

$\Rightarrow h - h/\sqrt{3} = 7$

$\Rightarrow h = \frac{7\sqrt{3}}{\sqrt{3}-1}$ m

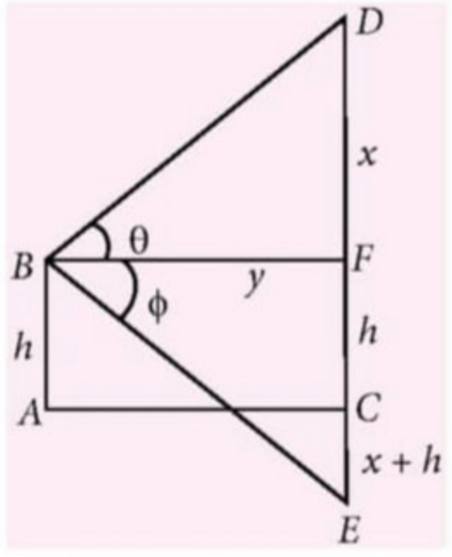


3. (a) : $\tan\theta = \frac{x}{y}$

$\tan\phi = \frac{2h + x}{y}$

$\Rightarrow x = \frac{2h}{\cot\theta \cdot \tan\phi - 1}$

$CD = h + x = \frac{h\sin(\phi+\theta)}{\sin(\phi-\theta)}$



4. (b) : $\frac{1+\tan\frac{y}{2}}{1-\tan\frac{y}{2}} = \left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)^3$

Squaring both sides, we get $\frac{1+\sin y}{1-\sin y} = \frac{(1+\sin x)^3}{(1-\sin x)^3}$

Using componendo and dividendo

$$\frac{2\sin y}{2} = \frac{(3+\sin^2 x)}{1+3\sin^2 x}\sin x$$

5. (b) : $\sin\alpha\,(\sec x_1\sec x_2 + \sec x_2\sec x_3 + ... + \sec x_{n-1}\sec x_n)$

$$= \frac{\sin(x_2 - x_1)}{\cos x_1\cos x_2} + \frac{\sin(x_3 - x_2)}{\cos x_2\cos x_3} + ... + \frac{\sin(x_n - x_{n-1})}{\cos x_{n-1}\cos x_n}$$

$= \tan x_2 - \tan x_1 + \tan x_3 - \tan x_2 + ... + \tan x_n - \tan x_{n-1}$

$$= \tan x_n - \tan x_1 = \frac{\sin(x_n - x_1)}{\cos x_n\cos x_1} = \frac{\sin(n-1)\alpha}{\cos x_n\cos x_1}$$

6. (a) : $a\tan^2 x + b = c(1 + \tan^2 x)$

$$\Rightarrow \tan^2 x = \left(\frac{c-b}{a-c}\right),\ \tan^2 y = \left(\frac{d-a}{b-d}\right)$$

$$\frac{a^2}{b^2} = \frac{\tan^2 y}{\tan^2 x} = \frac{(a-d)(c-a)}{(b-c)(d-b)}$$

7. (b) : $\cos^3 x\sin 2x = \cos^2 x\cos x\sin 2x$

$$= \left(\frac{1+\cos 2x}{2}\right)\left(\frac{2\sin 2x\cos x}{2}\right)$$

$$= \frac{1}{4}(1+\cos 2x)(\sin 3x + \sin x)$$

$$= \frac{1}{4}\left[\sin 3x + \sin x + \frac{1}{2}(2\sin 3x\cos 2x) + \frac{1}{2}(2\cos 2x\sin x)\right]$$

$$= \frac{1}{4}\left[\sin 3x + \sin x + \frac{1}{2}(\sin 5x + \sin x) + \frac{1}{2}(\sin 3x - \sin x)\right]$$

$$= \frac{1}{4}\left[\sin x + \frac{3}{2}\sin 3x + \frac{1}{2}\sin 5x\right]$$

By comparing, we get $n = 5$, $a_1 = \frac{1}{4}$

8. (a) : $\tan\theta/2 = \sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}$

$$= \sqrt{\frac{1-\left(\frac{a\cos\phi+b}{a+b\cos\phi}\right)}{1+\left(\frac{a\cos\phi+b}{a+b\cos\phi}\right)}} = \sqrt{\frac{(a-b)(1-\cos\phi)}{(a+b)(1+\cos\phi)}}$$

$$= \sqrt{\frac{a-b}{a+b}}\tan(\phi/2)$$

9. (d) : Let $I=\sum_{r=0}^{10}\frac{1}{4}\left(\cos 3\frac{\pi r}{3}+3\cos\frac{\pi r}{3}\right)$

$=\sum_{r=0}^{10}\frac{1}{4}\left(\cos\pi r+3\cos\frac{\pi r}{3}\right)=\frac{1}{4}(I_1+I_2)$

$\therefore\ I_1=\sum_{r=0}^{10}\cos\pi r=1-1+1-1+.....-1+1=1$

$$I_2=3\sum_{r=0}^{10}\cos\frac{\pi r}{3}=3\left[1+\frac{\cos\frac{11}{2}\cdot\frac{\pi}{3}\sin\frac{10\pi}{6}}{\sin\frac{\pi}{6}}\right]$$

$=3\left(1-\frac{3}{2}\right)=-\frac{3}{2}\Rightarrow I=\frac{1}{4}\left(1-\frac{3}{2}\right)=-\frac{1}{8}$

10. (d) : We have, $\sin^3 10^\circ+\sin^3 50^\circ-\sin^3 70^\circ$

$=\frac{1}{4}[(3\sin 10^\circ-\sin 30^\circ)+(3\sin 50^\circ-\sin 150^\circ)$

$-(3\sin 70^\circ-\sin 210^\circ)]$

$=\frac{1}{4}\left[3(\sin 10^\circ+\sin 50^\circ-\sin 70^\circ)-\frac{3}{2}\right]$

$=\frac{1}{4}\left[3(\sin 10^\circ-2\cos 60^\circ\cdot\sin 10^\circ)-\frac{3}{2}\right]=-\frac{3}{8}$

11. (a) : We have $\frac{\sin 2\beta}{3-\cos 2\beta}=\frac{2\sin\beta\cdot\cos\beta}{2-2\cos 2\beta+1+\cos 2\beta}$

$=\frac{2\sin\beta\cdot\cos\beta}{4\sin^2\beta+2\cos^2\beta}=\frac{\tan\beta}{1+2\tan^2\beta}=\frac{2\tan\beta-\tan\beta}{1+2\tan^2\beta}$ ...(i)

$\because\ \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$ ...(ii)

$\therefore$ By (i) and (ii), we get

$\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}=\frac{2\tan\beta-\tan\beta}{1+2\tan^2\beta}\Rightarrow \tan\alpha=2\tan\beta$

12. (b) : We have $A+B=\pi-C$

$\Rightarrow\ \tan(A+B)=-\tan C$

$\Rightarrow\frac{\tan A+\tan B}{1-\tan A\cdot\tan B}>0\ \left[\because \tan A>0,\tan B>0,\tan C<0\right]$

$=\tan A\cdot\tan B<1$

$\Rightarrow\ q<1$

13. (a) : Given, $2\sin x+2\sin^2 x-1=1$

or $\sin^2 x+\sin x-1=0$

$\therefore\ \sin x=\frac{-1\pm\sqrt{5}}{2}$

Consider, $\sin x=\frac{-1+\sqrt{5}}{2}$

$\Rightarrow\ \sin^2 x=\frac{3-\sqrt{5}}{2}\Rightarrow\cos^2 x=\frac{\sqrt{5}-1}{2}$

$\therefore\ \cos^2 x(1+\cos^2 x)=\frac{\sqrt{5}-1}{2}\times\frac{\sqrt{5}+1}{2}=1$

14. (b) : In a cyclic quadrilateral, no angle is greater than 180°

Here $\cos A=-\frac{12}{13}\Rightarrow\frac{\pi}{2}<A<\pi$ and $0<C<\pi/2$

(since $A+C=180^\circ$)

$\therefore\ \tan A=-\frac{5}{12}\ \Rightarrow\ \tan C=\frac{5}{12}$

Also $\tan B=\frac{4}{3}\Rightarrow 0<B<\frac{\pi}{2}$ and $\frac{\pi}{2}<D<\pi$

(since $B+D=180^\circ$)

$\therefore\ \cos B=\frac{3}{5}\ \Rightarrow\ \cos D=-\frac{3}{5}$

Now, the required equation is

$$x^2-\left(\frac{5}{12}-\frac{3}{5}\right)x+\left(\frac{5}{12}\right)\left(-\frac{3}{5}\right)=0$$

$\Rightarrow\ 60x^2+11x-15=0$

15. (a) : Given $\cot\frac{A}{2}\cdot\cot\frac{C}{2}=3$

$\Rightarrow\ \frac{\cos\frac{A}{2}\cdot\cos\frac{C}{2}}{\sin\frac{A}{2}\cdot\sin\frac{C}{2}}=3\ \Rightarrow\ \frac{\cos\frac{A-C}{2}}{\cos\frac{A+C}{2}}=2$

(using componendo and dividendo)

$\Rightarrow\ \frac{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}{2\sin\frac{A+C}{2}\cdot\cos\frac{A+C}{2}}=2\Rightarrow 2\sin B=\sin A+\sin C$

16. (b) : We have $\frac{2\tan\alpha}{1+\sec\alpha+\tan\alpha}=\frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}$

$=2\frac{2\tan\alpha/2}{(1+\tan^2\alpha/2)+(1-\tan^2\alpha/2)+2\tan\alpha/2}$

$=\frac{2\tan\alpha/2}{1+\tan\alpha/2}$

17. (a) : Since $A+B+C=\pi$

$\Rightarrow$ All of $\sin 3A$, $\sin 3B$, $\sin 3C$ can't be negative.

Let us take $\sin 3A=-1\ \Rightarrow\ A=\pi/2$

$\Rightarrow\ \sin 3A=-1$, $\sin 3B=-1$ and $\sin 3C=0$

So minimum value is -2.

18. (d) : $-\frac{\pi}{2} \le 2\tan^{-1}2x \le \frac{\pi}{2} \Rightarrow -\frac{1}{2} \le x \le \frac{1}{2}$

19. (b) : Let $\tan^{-1}x = \theta, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$3\theta + \cos^{-1}(\cos 3\theta) = 0$

$\cos^{-1}(\cos 3\theta) = -3\theta \Rightarrow -\pi \le 3\theta \le 0$

$\Rightarrow -\frac{\pi}{3} \le \theta \le 0$

$\Rightarrow x \in \left[-\sqrt{3}, 0\right]$

So, number of integral solutions is 2.

20. (c) : $a^2 + b^2 + c^2 = 4R^2(\sin^2A + \sin^2B + \sin^2C)$

$= 2R^2(1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C)$

$$= 2R^2\left[3 - \left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}\right)\right]$$

$$= 2R^2\left[3 - \left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right)\right]$$

$$= 2R^2\left[3 - \left(2\sin\frac{\pi}{7}\cdot\cos\frac{2\pi}{7} + 2\sin\frac{\pi}{7}\cdot\cos\frac{4\pi}{7} + 2\sin\frac{\pi}{7}\cdot\cos\frac{6\pi}{7}\right)\frac{1}{2\sin\frac{\pi}{7}}\right]$$

$$= 2R^2\left[3 - \frac{1}{2\sin\frac{\pi}{7}}\left(\sin\frac{3\pi}{7} - \sin\frac{\pi}{7} + \sin\frac{5\pi}{7} - \sin\frac{3\pi}{7} + \sin\frac{7\pi}{7} - \sin\frac{5\pi}{7}\right)\right]$$

$$= 2R^2\left[3 + \frac{1}{2}\right] = 7R^2$$

21. (d) : $\sin(\cot^{-1}(x+1)) = \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+2x+2}}\right)\right)$

$$\Rightarrow \sin(\cot^{-1}(x+1)) = \frac{1}{\sqrt{x^2+2x+2}} \quad \ldots(i)$$

and $\cos(\tan^{-1}x) = \cos\left(\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right) = \frac{1}{\sqrt{1+x^2}} \quad \ldots(ii)$

By (i) and (ii), we get $\frac{1}{x^2+2x+2} = \frac{1}{1+x^2} \Rightarrow x = -\frac{1}{2}$

22. (c) : $(x+3)^2 + 3 + 3\sin(a+bx) = 0$

$x = -3, \sin(a+bx) = -1$

$\Rightarrow \sin(a-3b) = -1$

$$a - 3b = (4n-1)\frac{\pi}{2}, n \in I$$

If $n = 1 \Rightarrow a - 3b = 3\pi/2 \Rightarrow \cos(a-3b) = 0$

23. (a) : Since, $\sum \cos A \operatorname{cosec} B \operatorname{cosec} C = \sum \frac{\cos A}{\sin B \sin C}$

$$= -\sum \frac{\cos(B+C)}{\sin B \sin C}$$

$$= \sum(1 - \cot B \cot C) = 3 - \sum \cot A \cot B = 2$$

24. (b) : $\tan^{-1}\left(\frac{\frac{a}{b+c} + \frac{b}{c+a}}{1 - \frac{a}{b+c}\cdot\frac{b}{c+a}}\right)$ as $\frac{ab}{(b+c)(c+a)} < 1$

But in right angled ΔABC

$c^2 = a^2 + b^2$

$\therefore$ Given expression $= \tan^{-1}(1) = \frac{\pi}{4}$

25. (a) : Given equations

$\Rightarrow 16 + 9 + 24\sin(P+Q) = 37$

$\Rightarrow P + Q = \frac{5\pi}{6}$ or $\frac{\pi}{6}$

If $P + Q = \frac{\pi}{6}$ then $R = \frac{5\pi}{6}$

If $P < \frac{\pi}{6}, 3\sin P < \frac{3}{2}$, then $3\sin P + 4\cos Q < \frac{3}{2} + 4 < 6$

$\therefore P + Q = \frac{\pi}{6}$ is not possible $\therefore R = \frac{\pi}{6}$

26. (a, b, c, d) : $\frac{\sin\theta}{\sin\phi}\frac{\sin\theta}{\sin\phi} = \frac{\sin\theta}{\sin\phi}\frac{\cos\phi}{\cos\theta}$

$$\Rightarrow \frac{\sin\theta}{\sin\phi} = \frac{\cos\phi}{\cos\theta} \Rightarrow \sin 2\theta = \sin 2\phi$$

$$\Rightarrow \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\tan\phi}{1+\tan^2\phi}$$

$$\Rightarrow \frac{6\tan\phi}{1+9\tan^2\phi} = \frac{2\tan\phi}{1+\tan^2\phi} \Rightarrow \tan^2\phi = \frac{1}{3} \Rightarrow \tan\theta = \pm\sqrt{3}$$

27. (a, b, c, d) : (a) $\sin\frac{11\pi}{12}\sin\frac{5\pi}{12}$

$$= \sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right) = \frac{1}{2}\sin\left(\frac{\pi}{6}\right) = \frac{1}{4} \in Q$$

(b) $\operatorname{cosec}\left(\frac{9\pi}{10}\right)\sec\left(\frac{4\pi}{5}\right) = -\operatorname{cosec}\left(\frac{\pi}{10}\right)\sec\left(\frac{\pi}{5}\right) = -4 \in Q$

(c) $1 - 2\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right) = 1 - \frac{1}{4} = \frac{3}{4} \in Q$

(d) $\left(2\cos^2\frac{\pi}{9}\right)\left(2\cos^2\frac{2\pi}{9}\right)\left(2\cos^2\frac{4\pi}{9}\right) = \frac{1}{8} \in Q$

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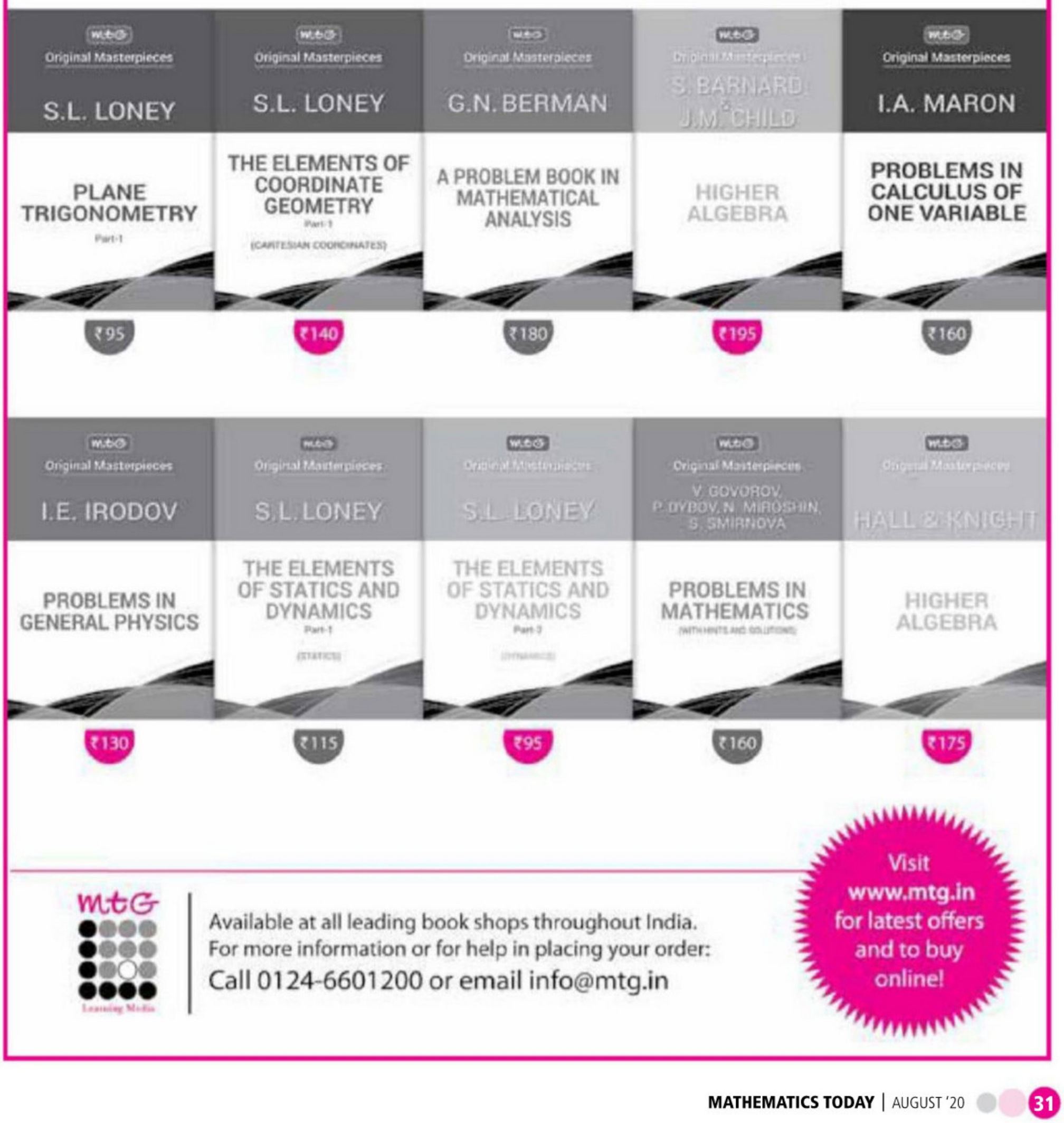

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28. (a, c, d) : $x - y = \frac{1}{3}$ and

$\cos\{\pi(x+y)\}\cos\{\pi(x-y)\} = \frac{1}{2}$

$\Rightarrow \quad x + y = 2n,\ n \in Z$

29. (a, b, c) : $2^{\text{cosec}^2 x}\sqrt{\frac{(y-1)^2+1}{2}} \le \sqrt{2}$

$\Rightarrow \quad 2^{\text{cosec}^2 x}\sqrt{(y-1)^2+1} \le 2$

$\Rightarrow$ $\text{cosec}^2 x = 1$ and $y = 1$

$\Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2}, y = 1$

30. (a, b, d) : α and β satisfy $x\cos\theta + y\sin\theta = 2a$

$\Rightarrow$ $(x^2 + y^2)\cos^2\theta - 4ax\cos\theta + (4a^2 - y^2) = 0$

$$\cos\alpha + \cos\beta = \frac{4ax}{x^2+y^2},\ \cos\alpha\cdot\cos\beta = \frac{4a^2-y^2}{x^2+y^2}$$

$$2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right) = 1 \Rightarrow 4\sin^2\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\beta}{2}\right) = 1$$

$\Rightarrow$ $\cos\alpha + \cos\beta = \cos\alpha\cdot\cos\beta$

$$\Rightarrow \quad \frac{4ax}{x^2+y^2} = \frac{4a^2-y^2}{x^2+y^2} \Rightarrow y^2 = 4a(a-x)$$

31. (a, b, c, d) : $\frac{k+1}{k-1} = 2\cos 2A$

$$\frac{\sin 3A}{\sin A} = 1 + 2\cos 2A = \frac{2k}{k-1}$$

$$\text{Also } k = \frac{3-\tan^2 A}{1-3\tan^2 A} \Rightarrow k < \frac{1}{3}, k > 3$$

32. (a, b, c, d) : $f_n(\theta) = \tan(\theta/2)\prod_{r=0}^{n}(1+\sec 2^r\theta)$

$$= \tan(\theta/2)\prod_{r=0}^{n}\left\{\frac{1+\cos(2^r\theta)}{\cos(2^r\theta)}\right\}$$

$$= \tan(\theta/2)\prod_{r=0}^{n}\frac{2\cos^2(2^{r-1}\theta)}{\cos(2^r\theta)}$$

$$= 2^{n+1}\cdot\tan(\theta/2)\prod_{r=0}^{n}\frac{\cos^2(2^{r-1}\theta)}{\cos(2^r\theta)}$$

$$= 2^{n+1}\cdot\tan(\theta/2)\cdot\cos^2(\theta/2)\prod_{r=1}^{n}\frac{\cos(2^{r-1}\theta)}{\cos(2^n\theta)}$$

$$= 2^n\cdot\sin\theta\cdot\frac{\sin(2^n\theta)}{2^n.\sin\theta.\cos(2^n\theta)} = \tan(2^n\theta)$$

$\therefore$ Alternate (a) : $f_2\left(\frac{\pi}{16}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

Alternate (b) : $f_3\left(\frac{\pi}{32}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

Alternate (c) : $f_4\left(\frac{\pi}{64}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

Alternate (d) : $f_5\left(\frac{\pi}{128}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

33. (a, b) : $T_r = \sin^{-1}\left(\frac{\sqrt{r^2+2r} - \sqrt{r^2-1}}{r(r+1)}\right)$

$$T_r = \sin^{-1}\left(\frac{1}{r}\sqrt{1-\frac{1}{(r+1)^2}} - \frac{1}{r+1}\sqrt{1-\frac{1}{r^2}}\right)$$

$$T_r = \sin^{-1}\left(\frac{1}{r}\right) - \sin^{-1}\left(\frac{1}{r+1}\right)$$

$$\Rightarrow S_n = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{n+1}\right) = \cos^{-1}\left(\frac{1}{n+1}\right)$$

34. (a, b) : $\sec\left(\frac{x+y}{2}\right) \le \frac{\sec x + \sec y}{2}$

(35-37) : 35. (b) 36. (b) 37. (d)

35. (b) : $\sec\frac{\pi}{7}, \sec\frac{3\pi}{7}, \sec\frac{5\pi}{7}$ are the roots of

$x^3 - 4x^2 - 4x + 8 = 0$

$$\sec\left(\frac{\pi}{7}\right)+\sec\left(\frac{3\pi}{7}\right)+\sec\left(\frac{5\pi}{7}\right)=4$$

$$8x^3-4x^2-4x+1=8\left(x-\cos\frac{\pi}{7}\right)\left(x-\cos\frac{3\pi}{7}\right)\left(x-\cos\frac{5\pi}{7}\right)$$

36. (b) : Put $x = 1$

$$\Rightarrow \sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)=\frac{1}{8}$$

37. (d) : Put $x = -1$

$$\Rightarrow \cos\left(\frac{\pi}{14}\right)\cos\left(\frac{3\pi}{14}\right)\cos\left(\frac{5\pi}{14}\right)=\frac{\sqrt{7}}{8}$$

38. A - p; B - p; C - r; D - s

(A) $\tan^2\theta = \sin\theta\cos\theta \Rightarrow \sin\theta = \cos^3\theta$

$\therefore$ $(1 - \sin^2\theta) + (1 - 3\sin^2\theta) + 3\sin^4\theta - \sin^6\theta$

$= \cos^2\theta + (1 - \sin^2\theta)^3 = \cos^2\theta + \cos^6\theta = \cos^2\theta + \sin^2\theta = 1$

(B) $\sin40° = \sin(60° - 20°)$

$$\Rightarrow 2\sin20°\cos20°=\frac{\sqrt{3}}{2}\cos20°-\frac{1}{2}\sin20°$$

$$\Rightarrow 4\cos20°=\sqrt{3}\cot20°-1$$

$$\text{(C) } \frac{3+\cot76°\cot16°}{\cot76°+\cot16°}=\frac{3\sin76°\sin16°+\cos76°\cos16°}{\sin(76°+16°)}$$

$$=\frac{2\sin76°\sin16°+\cos(76°-16°)}{\sin(76°+16°)}$$

$$=\frac{\cos60°-\cos92°+\cos60°}{\sin92°}=\frac{1-\cos92°}{\sin92°}=\tan46°$$

$= \cot 44°$

$$\text{(D) } \sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{2\pi}{18}\right)+....+\sin^2\left(\frac{\pi}{2}\right)=5$$

39. A - r; B - s; C - p; D - q

(A) Let $A=20°\Rightarrow 3A=60°\Rightarrow \cos3A=\frac{1}{2}$

$\Rightarrow 4\cos^3 A-3\cos A=\frac{1}{2} \Rightarrow 8x^3 - 6x - 1 = 0,$

where $x = \cos 20°$

(B) Let $A=10°\Rightarrow \sin3A=\frac{1}{2}\Rightarrow 8x^3-6x+1=0,$

where $x = \sin10°$

(C) Let $A = 15° \Rightarrow 3A = 45°$

$$\Rightarrow \tan3A = 1 \Rightarrow \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}=1$$

$\Rightarrow x^3 - 3x^2 - 3x + 1 = 0$, where $x = \tan15°$

(D) Let $A=6°\Rightarrow \sin5A=\frac{1}{2}\Rightarrow 32x^5-40x^3+10x-1=0,$

where $x = \sin6°$

40. A - p; B - q; C - s; D - q, r

(A) Given, $\sin\theta = 3\sin(\theta + 2\alpha)$

$\Rightarrow \sin(\theta + \alpha - \alpha) = 3\sin(\theta + \alpha + \alpha)$

$\Rightarrow \sin(\theta + \alpha)\cos\alpha - \cos(\theta + \alpha)\sin\alpha$

$= 3\sin(\theta + \alpha)\cos\alpha + 3\cos(\theta + \alpha)\sin\alpha$

$\Rightarrow -2\sin(\theta + \alpha)\cos\alpha = 4\cos(\theta + \alpha)\sin\alpha$

$$\Rightarrow -\frac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)}=\frac{2\sin\alpha}{\cos\alpha} \Rightarrow -\tan(\theta + \alpha) = 2\tan\alpha$$

$\Rightarrow \tan(\theta + \alpha) + 2\tan\alpha = 0$

(B) We have, $p\sin\theta + q\cos\theta = a$ ...(1)

and $p\cos\theta - q\sin\theta = b$ ...(2)

Squaring (1) and (2) and then adding, we get

$(p\sin\theta + q\cos\theta)^2 + (p\cos\theta - q\sin\theta)^2 = a^2 + b^2$

$\Rightarrow p^2(1) + q^2(1) - a^2 - b^2 = 0$

$\Rightarrow (p^2 - a^2) + (q^2 - b^2) = 0$

$\Rightarrow (p + a)(p - a) + (q + b)(q - b) = 0$

$$\Rightarrow \frac{p+a}{q+b}+\frac{q-b}{p-a}=0$$

(C) Consider, $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$

$$=\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)$$

$$=\cos\left(\frac{3\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)$$

$$=-\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

Also, $\cos\frac{10\pi}{7}=\cos\frac{4\pi}{7}$

So, $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{10\pi}{7}-\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$

$$=2\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)=-\frac{1}{4}$$

(D) Clearly, $\sec\theta - \tan\theta = 1$

Also, 1 satisfy the given equation.

So, the roots of the given equation are 1 & $\sec\theta$.

41. (0) : Let $\theta=\frac{\pi}{18} \Rightarrow 6\theta=\frac{\pi}{3} \Rightarrow \cos6\theta=\frac{1}{2}$

$$\Rightarrow 4\cos^3 2\theta-3\cos2\theta=\frac{1}{2}$$

$\Rightarrow 8(2\cos^2\theta-1)^3-6(2\cos^2\theta-1)=1.$

Let $2\cos\theta = x$

$$\Rightarrow\ 8\left(2.\frac{x^2}{4}-1\right)^3-6\left(2.\frac{x^2}{4}-1\right)=1$$

$\Rightarrow\ (x^2-2)^3-3(x^2-2)=1 \Rightarrow x^6-6x^4+9x^2-3=0$

$f'(x)=6x\,(x^4-4x^2+3)$

$f'(x)=0 \Rightarrow x=0,\pm1,\pm\sqrt{3}$

42. (0) : $\cos\theta\,(1+\cos^2\theta)=\sin^2\theta$

$(1-\sin^2\theta)\,(2-\sin^2\theta)^2=\sin^4\theta$

$\sin^6\theta=4-8\sin^2\theta+4\sin^4\theta$

On comparing with the given equation, we get

$a=4,\ b=-8,\ c=4$

$\therefore\ a+b+c=0$

43. (0) : $\dfrac{\sin\theta}{\cos3\theta}=\dfrac{2\sin\theta\cos\theta}{2\cos3\theta\cos\theta}$

$=\dfrac{\sin2\theta}{2\cos3\theta\cos\theta}=\dfrac{\sin(3\theta-\theta)}{2\cos3\theta\cos\theta}$

$\therefore\ \dfrac{\sin\theta}{\cos3\theta}=\dfrac{1}{2}[\tan3\theta-\tan\theta]$ ...(1)

$\dfrac{\sin3\theta}{\cos9\theta}=\dfrac{1}{2}[\tan9\theta-\tan3\theta]$ ...(2)

$\dfrac{\sin9\theta}{\cos27\theta}=\dfrac{1}{2}[\tan27\theta-\tan9\theta]$ ...(3)

Adding (1), (2) and (3), we get

$$\frac{\sin\theta}{\cos3\theta}+\frac{\sin3\theta}{\cos9\theta}+\frac{\sin9\theta}{\cos27\theta}=\frac{1}{2}[\tan27\theta-\tan\theta]$$

Now, on comparing, we get

$A=2,\ B=27,\ C=1$

$\therefore\ 27A-B-27C=0$

44. (3) : Let $\tan x=2t,\ \tan y=3t,\ \tan z=5t$

$$\sum\tan x=\tan x\cdot\tan y\cdot\tan z\Rightarrow t^2=\frac{1}{3}$$

Now, $\tan^2x+\tan^2y+\tan^2z=t^2\,(4+9+25)$

$=38t^2,\ K=3$

45. (1) : Let $A=\dfrac{\beta+\gamma-\alpha}{4}, B=\dfrac{\gamma+\alpha-\beta}{4}$ and

$C=\dfrac{\alpha+\beta-\gamma}{4}$

$\Rightarrow\ \tan A\,\tan B\,\tan C=1$

$\Rightarrow\ \dfrac{\sin A\sin B}{\cos A\cos B}=\dfrac{1}{\tan C}$

$\Rightarrow\ \dfrac{\sin A\sin B-\cos A\cos B}{\sin A\sin B+\cos A\cos B}=\dfrac{1-\tan C}{1+\tan C}$

[By Componendo and Dividendo]

$$\Rightarrow\frac{-\cos(A+B)}{\cos(A-B)}=\frac{\sin\left(\frac{\pi}{4}-C\right)}{\cos\left(\frac{\pi}{4}-C\right)}$$

$$\Rightarrow 2\sin\left(\frac{\pi}{4}-C\right)\cos(A-B)+2\cos\left(\frac{\pi}{4}-C\right)\cos(A+B)=0$$

$$\Rightarrow\sin\left(\frac{\pi}{4}-C+A-B\right)+\sin\left(\frac{\pi}{4}-C-A+B\right)$$

$$+\cos\left(\frac{\pi}{4}-C+A+B\right)+\cos\left(\frac{\pi}{4}-C-A-B\right)=0\ \ ...(1)$$

$A-B-C=\dfrac{\pi}{4}-\alpha,\ B-A-C=\dfrac{\pi}{4}-\beta,$

$C-A-B=\dfrac{\pi}{4}-\gamma$ and $A+B+C=\dfrac{\pi}{4}$

$\therefore\ (1)\ \Rightarrow\ \cos\alpha+\cos\beta+\cos\gamma+1=0$

Thus, $K=1$.

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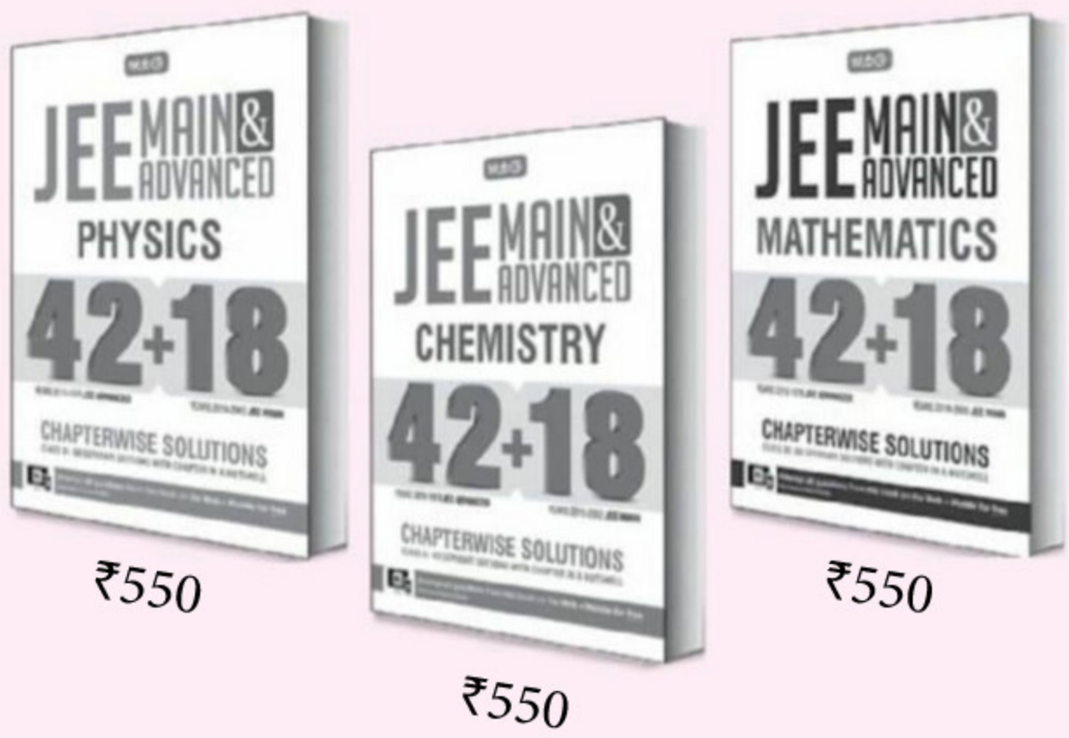

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CBSE warm-up!

CLASS-XI

Chapterwise Practice questions for CBSE Exams as per the reduced syllabus and marking scheme issued by CBSE for the academic session 2020-21.

Series 3 Complex Numbers and Quadratic Equations

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) This question paper contains **36** questions.
(iii) Question **1-20** in Section-A are very short-answer-objective type questions carrying **1** mark each.
(iv) Question **21-26** in Section-B are short answer type questions carrying **2** marks each.
(v) Question **27-32** in Section-C are long answer-**I** type questions carrying **4** marks each.
(vi) Question **33-36** in Section-D are long answer-**II** type questions carrying **6** marks each.

SECTION-A

(Q.1 - Q.10) are multiple choice type questions. Select the correct option.

1. Express in standard form : $(1 + 2i)^6(4i + 3)^3$
(a) 15625 (b) −15625
(c) 14625 (d) −14625

2. Evaluate : $\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2$
(a) −4 (b) −3 (c) −2 (d) −5

3. Find the multiplicative inverse of $\sqrt{5} + 3i$.
(a) $-\frac{\sqrt{5}}{14} - \frac{3}{14}i$ (b) $\frac{\sqrt{5}}{14} - \frac{3}{14}i$
(c) $\frac{\sqrt{5}}{14} + \frac{3}{14}i$ (d) $-\frac{\sqrt{5}}{14} + \frac{3}{14}i$

4. Write the principal argument of $z = 3 + i\sqrt{3}$.
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

5. Let Z be a complex number satisfying the equation $(Z^3 + 3)^2 = -16$, then $|Z|$ has the value equal to
(a) $5^{1/2}$ (b) $5^{1/3}$ (c) $5^{2/3}$ (d) 5

6. Evaluate : $i^{141} + i^{142} + i^{143} + i^{144}$
(a) −1 (b) 0 (c) 1 (d) 2

7. Solve : $x^2 - 14x + 58 = 0$.
(a) $\sqrt{7} \pm 3i$ (b) $7 \pm \sqrt{3}i$
(c) $7 \pm 3i$ (d) $\sqrt{7} \pm \sqrt{3}i$

8. Solve : $3x^2 + 8ix + 3 = 0$.
(a) $i/3$ (b) $-3i$
(c) Both (a) and (b) (d) None of these

9. On multiplying $3 - 2i$ by its conjugate, we get _____ .
(a) 10 (b) 11 (c) 12 (d) 13

10. Find the multiplicative inverse of $\frac{3+4i}{4-5i}$.
(a) $\frac{8}{25} + i\frac{31}{25}$ (b) $\frac{-8}{25} + i\frac{31}{25}$
(c) $\frac{-8}{25} - i\frac{31}{25}$ (d) $\frac{8}{25} - i\frac{31}{25}$

(Q. 11-Q.15) Fill in the blanks.

11. The sum of the series $i + i^2 + i^3 + ...$ upto 1000 terms is _____ .

12. The value of $\sqrt{-36} \times \sqrt{-16}$ is _____ .

OR

For any complex number z, the minimum value of $|z| + |z-1|$ is _____ .

13. If z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then arg $(z_1 - z_2)$ = _____ .

14. The number $\frac{(1-i)^3}{1-i^3}$ is equal to _____ .

OR

Multiplicative inverse of $1 - i$ is _____ .

15. The amplitude of $-1+\sqrt{-3}$ is _____ .

(Q. 16-Q.20) Answer the following questions.

16. Find the additive inverse of $\frac{3}{\sqrt{2}+i}$.

17. Show that for every complex number $z = x + iy \neq 0 + i \cdot 0$; there exists a complex number $z_1 = \frac{x-iy}{x^2+y^2}$ such that $zz_1 = 1$

OR

Write the complex number $z = \frac{2+i}{(1+i)(1-2i)}$ in standard form.

18. Solve the following equation by factorization method : $x^2 - ix + 6 = 0$.

19. Find the modulus of $\frac{1+i}{1-i}$.

20. Solve $x^2 + 3 = 0$.

SECTION-B

21. Show that the points representing the complex numbers $(3 + 2i)$, $(2 - i)$ and $-7i$ are collinear.

22. Find the argument of $\frac{1+i}{1-i}$.

OR

What are the real numbers 'x' and 'y', if $(x - iy)(3 + 5i)$ is the conjugate of $(-1 - 3i)$?

23. Express $(4 - 3i)^3$ in the standard form.

24. Solve the equation $x^2 - (3\sqrt{2}+2i)x + 6\sqrt{2}\,i = 0$ using factorization method.

25. Express $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}$ in the form $a + ib$.

OR

Find the solution of equation $x^2 + x + 1 = 0$.

26. If $\frac{(a^2+1)^2}{2a-i} = x + iy$, then find the value of $x^2 + y^2$.

SECTION-C

27. Show that : $\frac{(1+i)(3+i)}{(3-i)} - \frac{(1-i)(3-i)}{(3+i)} = \frac{14}{5}i$

28. Solve : $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

OR

If $(1 + 2i)(2 + 3i)(3 + 4i) = x + iy$, then prove that $x^2 + y^2 = 1625$.

29. If $z_1 = 2 - i$, $z_2 = -2 + i$, then find $\text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$.

30. Find the conjugate and argument of $\frac{2+i}{4i+(1+i)^2}$.

OR

Find x and y if $(x^4 + 2xi) - (3x^2 + iy) = (3 - 5i) + (1 + 2iy)$.

31. Find the modulus and the principal argument of the complex number $\frac{1-i}{3+i} + 4i$.

32. For all $z_1, z_2, \in C$, prove that:

(i) $\overline{(z_1 + z_2)} = (\bar{z}_1 + \bar{z}_2)$ (ii) $\overline{(z_1 - z_2)} = (\bar{z}_1 - \bar{z}_2)$

SECTION-D

33. Reduce $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$ to the standard form.

34. For complex values of z, solve $|z| + z = (2 + i)$.

OR

If $a + ib = \frac{c+i}{c-i}; a, b, c \in R$, then show that

$a^2 + b^2 = 1$ and $\frac{b}{a} = \frac{2c}{c^2-1}$.

35. If $|z_1| = |z_2| = |z_3| = ... = |z_n| = 1$, then prove that

$$\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + ... + \frac{1}{z_n}\right| = |z_1 + z_2 + z_3 + ... + z_n|$$

OR

Solve the equation : $2|z|^2 + z^2 - 5 + i\sqrt{3} = 0$

36. If $f(z)$ is a polynomial in z such that when it is divided by $z + 2i$ and $z - 2i$, the remainders are respectively $1 + i$ and $1 - i$. Determine the remainder when $f(z)$ is divided by $z^2 + 4$.

SOLUTIONS

1. (b) : We have, $(1 + 2i)^6(4i + 3)^3$

$= ((1 + 2i)^3)^2(4i + 3)^3 = (1 + 8i^3 + 6i + 12i^2)^2$

$(64i^3 + 27 + 108i + 144i^2)$

$= (1 - 12 + 6i - 8i)^2 (-64i + 108i + 27 - 144)$

$= (-11 - 2i)^2(44i - 117) = 1936i^2 - 13689 = -15625$

2. (a) : Given expression

$$\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2 = \left[i^{18} \cdot i + \left(\frac{1}{i}\right)^{25}\right]^2$$

$= [(i^2)^9 \cdot i + (-i)^{25}]^2 = [(-1)^9 \cdot i - i^{25}]^2 = [-i - i^{24} \cdot i]^2$

$= [-i - (i^8)^3 \cdot i]^2 = [-i - (1)^3 i]^2$

$= (-i - i)^2 = (-2i)^2 = 4i^2 = -4$

3. (b) : Let $z = \sqrt{5} + 3i$

$\bar{z} = \sqrt{5} - 3i$ and $|z|^2 = (\sqrt{5})^2 + (3)^2 = 14$

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5} - 3i}{14} = \frac{\sqrt{5}}{14} - \frac{3}{14}i$$

4. (b) : Let $z = 3 + i\sqrt{3}$

$$\therefore \quad \arg(z) = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

5. (b) : $(Z^3 + 3)^2 = 16i^2$

$\Rightarrow \quad Z^3 + 3 = 4i \quad$ or $\quad -4i$

$\Rightarrow \quad Z^3 = -3 + 4i \quad$ or $\quad -3 - 4i$

$\Rightarrow \quad |Z|^3 = |-3 + 4i| = 5 \quad \therefore \quad |Z| = 5^{1/3}$

6. (b) : We have, $i^{141} + i^{142} + i^{143} + i^{144}$

$= i^{140}[i + i^2 + i^3 + i^4]$

$= (i^4)^{35}\,[i - 1 - i + 1] \qquad [\because i^2 = -1, i^3 = -i, i^4 = 1]$

$= 0.$

7. (c) : By Quadratic formula, we have

$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(58)}}{2}$$

$$= \frac{14 \pm \sqrt{196 - 232}}{2} = \frac{14 \pm 6i}{2} = 7 \pm 3i$$

8. (c) : Using Quadratic formula, we have

$$x = \frac{-8i \pm \sqrt{(8i)^2 - 4 \times 3 \times 3}}{2 \times 3} = \frac{-8i \pm \sqrt{-100}}{6} = \frac{-8i \pm 10i}{6}$$

$$\Rightarrow \quad x = \frac{i}{3} \text{ or } -3i$$

9. (d) : Let $z = 3 - 2i$

$\bar{z} = 3 + 2i$

$\therefore \quad z\bar{z} = (3 - 2i)(3 + 2i) = 9 - 4i^2 = 9 + 4 = 13$

10. (c) : Multiplicative inverse of $\frac{3+4i}{4-5i}$

$$= \frac{4-5i}{3+4i} = \frac{4-5i}{3+4i} \times \frac{3-4i}{3-4i}$$

$$= \frac{(12-20) + i(-15-16)}{9+16} = -\frac{8}{25} - i\frac{31}{25}$$

11. Let $S = i + i^2 + i^3 + \ldots\ldots$ upto 1000 terms

$= (i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + \ldots\ldots$ 250 brackets

$= 0 + 0 + 0 + \ldots\ldots + 0$

$(\because \; i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$, where $n \in N)$

$= 0$

12. We have, $\sqrt{-36} \times \sqrt{-16} = 6i \times 4i = i^2(6 \times 4) = -24$

OR

We know that $|z_1| + |z_2| \geq |z_1 - z_2|$

$\Rightarrow \quad |z| + |z-1| \geq |z-(z-1)|$

$\Rightarrow \quad |z| + |z-1| \geq 1$

So, minimum value of $|z| + |z-1|$ is 1.

13. Given, $|z_1 + z_2| = |z_1| + |z_2|$

$\Rightarrow \quad |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2|$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2\text{Re}\,(z_1\bar{z}_2) = |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2|$

$\Rightarrow \quad 2\text{Re}\,(z_1\bar{z}_2) = 2\,|z_1|\,|z_2|$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 1 \Rightarrow \theta_1 - \theta_2 = 0$

$\Rightarrow \quad \arg(z_1) - \arg(z_2) = 0$

14. $$\frac{(1-i)^3}{1-i^3} = \frac{(1-i)^3}{(1-i)(1+i+i^2)}$$

$$= \frac{(1-i)^2}{i} = \frac{1+i^2-2i}{i} = \frac{-2i}{i} = -2$$

OR

Multiplicative inverse of $1 - i = \frac{1}{1-i}$

$$= \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{1-i^2} = \frac{1}{2}(1+i)$$

15. We have, $z = -1 + \sqrt{-3} = -1 + i\sqrt{3}$

$\therefore \quad \arg(z) = \text{amp}(z)$

$$= \pi - \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \pi - \tan^{-1}\sqrt{3} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

16. Let $z = \frac{3}{\sqrt{2}+i} = \frac{3(\sqrt{2}-i)}{2+1} = \sqrt{2} - i$

$\therefore \quad$ Additive inverse of z is $-z$ *i.e.* $-\sqrt{2} + i$

17. L.H.S. $= zz_1 = (x+iy)\left(\frac{x-iy}{x^2+y^2}\right)$

$$= (x+iy)\left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right)$$

$$= \left(\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}\right)+i\left(\frac{-xy}{x^2+y^2}+\frac{xy}{x^2+y^2}\right)$$
$= 1 + i\,.\,(0) = 1 =$ R.H.S.

OR

We have, $z = \dfrac{2+i}{(1+i)(1-2i)} = \dfrac{2+i}{3-i} = \dfrac{(2+i)(3+i)}{(3-i)(3+i)}$

$= \dfrac{5+5i}{10} = \dfrac{1}{2}+i\dfrac{1}{2}$

18. Given, $x^2 - ix + 6 = 0$

$\Rightarrow \quad x^2 - 3ix + 2ix + 6 = 0 \quad \Rightarrow x(x - 3i) + 2i(x - 3i) = 0$

$\Rightarrow \quad (x + 2i)(x - 3i) = 0 \quad \Rightarrow x = -2i$ or $x = 3i$

Hence, the roots of the given equation are $-2i$ and $3i$.

19. We have, $\left|\dfrac{1+i}{1-i}\right| = \dfrac{|1+i|}{|1-i|} = \dfrac{\sqrt{1+1}}{\sqrt{1+1}} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$

20. (b) : Given, $x^2 + 3 = 0 \Rightarrow x^2 = -3$

$\Rightarrow \quad x = \pm\sqrt{-3} = \pm\sqrt{3}i$

21. Let these numbers be represented on the argand plane by the points A, B and C respectively.

Then, $AB = |(2 - i) - (3 + 2i)| = |-1 - 3i| = \sqrt{10}$

$BC = |-7i - (2 - i)| = |-2 - 6i| = \sqrt{40} = 2\sqrt{10}$

and $AC = |-7i - (3+2i)| = |-3 - 9i| = \sqrt{90} = 3\sqrt{10}$

Clearly, $AB + BC = AC$

Hence, the points A, B, C are collinear.

22. We have,

$\dfrac{1+i}{1-i} = \dfrac{(1+i)(1+i)}{(1-i)(1+i)} = \dfrac{(1+i)^2}{1+1} = \dfrac{1-1+2i}{2}$

$= i = 0 + 1i$

Now, $\tan\alpha = \dfrac{1}{0} = \infty$

$\Rightarrow \quad \alpha = \dfrac{\pi}{2}$

Argument $= \dfrac{\pi}{2}$

OR

Given, $(x-iy)(3+5i) = \overline{-1-3i}$

$\Rightarrow x - iy = \dfrac{-1+3i}{3+5i} = \dfrac{(-1+3i)(3-5i)}{9-25i^2}$

$\Rightarrow x - iy = \dfrac{-3+5i+9i-15i^2}{9+25} = \dfrac{12+14i}{34} = \dfrac{6}{17}+\dfrac{7}{17}i$

$\Rightarrow x = \dfrac{6}{17}, y = -\dfrac{7}{17}$

23. We have, $(4 - 3i)^3 = 4^3 - (3i)^3 - 3 \times 4 \times 3i \times (4 - 3i)$

$= 64 - 27i^3 - 36i(4 - 3i) = (64 - 108) + i(27 - 144)$

$= -44 - 117i$

24. We have, $x^2 - (3\sqrt{2}+2i)x + 6\sqrt{2}\,i = 0$

$\Rightarrow \quad (x^2 - 3\sqrt{2}x) - 2ix + 6\sqrt{2}\,i = 0$

$\Rightarrow \quad x(x - 3\sqrt{2}) - 2i(x - 3\sqrt{2}) = 0$

$\Rightarrow \quad (x - 2i)(x - 3\sqrt{2}) = 0$

$\Rightarrow \quad x = 2i$ or $x = 3\sqrt{2}$

Hence, the roots of the given equation are $2i$ and $3\sqrt{2}$.

25. We have, $\dfrac{5+\sqrt{2}i}{1-\sqrt{2}i}$

$= \dfrac{(5+\sqrt{2}i)(1+\sqrt{2}i)}{(1-\sqrt{2}i)(1+\sqrt{2}i)} = \dfrac{5+5\sqrt{2}i+\sqrt{2}i+2i^2}{(1)^2-(\sqrt{2}i)^2}$

$= \dfrac{5+6\sqrt{2}i-2}{1-2i^2} = \dfrac{3+6\sqrt{2}i}{1+2} = \dfrac{3+6\sqrt{2}i}{3} = 1+2\sqrt{2}i$

OR

Given, $x^2 + x + 1 = 0$

$\therefore \quad x = \dfrac{-1\pm\sqrt{(1)^2-4\times1\times1}}{2\times1} = \dfrac{-1\pm\sqrt{-3}}{2}$

$= \dfrac{-1\pm\sqrt{3}i}{2} = \dfrac{-1}{2}\pm\dfrac{\sqrt{3}i}{2}$

26. We have, $x + iy = \dfrac{(a^2+1)^2}{2a-i}$ ...(i)

Changing i to $-i$, we get

$x - iy = \dfrac{(a^2+1)^2}{2a+i}$ ...(ii)

Multiplying (i) and (ii), we get

$(x + iy)(x - iy) = \dfrac{(a^2+1)^2}{2a-i} \times \dfrac{(a^2+1)^2}{2a+i}$

$\Rightarrow \quad x^2 + y^2 = \dfrac{(a^2+1)^4}{4a^2+1}$

27. We have, $\dfrac{(1+i)(3+i)}{(3-i)} = \dfrac{(3-1)+(3+1)i}{(3-i)}$

$= \dfrac{(2+4i)}{(3-i)} = \dfrac{(2+4i)}{(3-i)} \times \dfrac{(3+i)}{(3+i)}$

$= \dfrac{(2+4i)(3+i)}{(9+1)} = \dfrac{(6-4)+(12+2)i}{10} = \dfrac{(2+14i)}{10} = \dfrac{(1+7i)}{5}$

Also, $\dfrac{(1-i)(3-i)}{(3+i)} = \dfrac{(3-1)-4i}{(3+i)}$

$= \dfrac{(2-4i)}{(3+i)} \times \dfrac{(3-i)}{(3-i)} = \dfrac{(6-4)-14i}{(9+1)} = \dfrac{(2-14i)}{10} = \dfrac{(1-7i)}{5}$

Now, L.H.S. of given expression $= \dfrac{(1+7i)}{5} - \dfrac{(1-7i)}{5}$

$= \dfrac{(1+7i)-(1-7i)}{5} = \dfrac{14}{5}i$ = R.H.S.

28. The given equation is of the form $ax^2 + bx + c = 0$

Here, $a = \sqrt{3}$, $b = -\sqrt{2}$ and $c = 3\sqrt{3}$

$\therefore$ The roots are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\sqrt{2} \pm \sqrt{(-\sqrt{2})^2 - 4 \times \sqrt{3} \times 3\sqrt{3}}}{(2 \times \sqrt{3})}$$

$$= \frac{\sqrt{2} \pm \sqrt{2 - 36}}{2\sqrt{3}} = \frac{\sqrt{2} \pm \sqrt{-34}}{2\sqrt{3}} = \frac{\sqrt{2} \pm i\sqrt{34}}{2\sqrt{3}}$$

Hence, the roots of the given equation are

$$\frac{\sqrt{2} + i\sqrt{34}}{2\sqrt{3}} \text{ and } \frac{\sqrt{2} - i\sqrt{34}}{2\sqrt{3}}.$$

OR

We have, $x + iy = (1 + 2i)(2 + 3i)(3 + 4i)$

$= (2 + 3i + 4i + 6i^2)(3 + 4i) = (-4 + 7i)(3 + 4i)$

$= -12 - 16i + 21i + 28i^2 = -40 + 5i$

Equating the real and the imaginary parts, we get

$x = -40, y = 5$

Now, $x^2 + y^2 = (-40)^2 + (5)^2 = 1600 + 25 = 1625$

29. We have, $z_1 = 2 - i, z_2 = -2 + i$

$z_1 z_2 = (2 - i)(-2 + i) = (-4 + 2i + 2i - i^2)$

$= -3 + 4i$

Also, $\overline{z}_1 = 2 + i$

$$\Rightarrow \quad \frac{1}{\overline{z}_1} = \frac{1}{2+i} = \frac{2-i}{4-i^2} = \frac{2-i}{5}$$

$$\text{Now, } \frac{z_1 z_2}{\overline{z}_1} = (-3 + 4i)\left(\frac{2-i}{5}\right) = \frac{1}{5}(-6 + 3i + 8i - 4i^2)$$

$$= \left(\frac{-2 + 11i}{5}\right)$$

$$\therefore \quad \text{Re}\left(\frac{z_1 z_2}{\overline{z}_1}\right) = -\frac{2}{5}$$

30. Let $z = \frac{2+i}{4i + (1+i)^2} = \frac{2+i}{4i + 1 + i^2 + 2i} = \frac{2+i}{6i}$

$$= \left(\frac{2+i}{6i}\right)\left(\frac{-6i}{-6i}\right) = \frac{6 - 12i}{36} = \frac{1}{6} - \frac{1}{3}i$$

$$\therefore \quad \overline{z} = \frac{1}{6} + \frac{1}{3}i$$

$$\text{Now } z = \frac{1}{6} - \frac{1}{3}i$$

$$\therefore \quad x = \frac{1}{6}, y = -\frac{1}{3}$$

$$\tan\theta = \left|\frac{y}{x}\right| = \left|\frac{-1/3}{1/6}\right| = |-2| = 2$$

$$\therefore \quad \theta = \tan^{-1} 2, 0 \le \theta < \frac{\pi}{2}$$

$$x = \frac{1}{6} > 0 \text{ and } y = -\frac{1}{3} < 0.$$

Hence z lies in the 4th quadrant.

$\therefore$ $\arg z = 2\pi - \theta = 2\pi - \tan^{-1} 2$

OR

We have, $(x^4 + 2xi) - (3x^2 + iy) = (3 - 5i) + (1 + 2iy)$

$\Rightarrow$ $(x^4 - 3x^2) + (2x - y)i = 4 + (-5 + 2y)i$

Equating real and imaginary parts, we get

$x^4 - 3x^2 - 4 = 0$ ...(i)

and $2x - 3y + 5 = 0$ ...(ii)

Now, $x^4 - 3x^2 - 4 = 0 \Rightarrow (x^2 - 4)(x^2 + 1) = 0$

$\Rightarrow$ $x^2 - 4 = 0$ [$\because x^2 + 1 \neq 0$ for any real x]

$\Rightarrow$ $x = \pm 2$

Putting $x = \pm 2$ in (ii), we get

$y = 3$ when $x = 2$ and $y = \frac{1}{3}$ when $x = -2$.

31. Let $z = \frac{1-i}{3+i} + 4i = \frac{1 - i + 12i + 4i^2}{3+i}$

$$= \frac{1 - i + 12i - 4}{3+i} = \frac{-3 + 11i}{3+i}$$

$$= \frac{-3 + 11i}{3+i} \times \frac{3-i}{3-i} = \frac{-9 + 3i + 33i - 11i^2}{9 - i^2}$$

$$\Rightarrow \quad z = \frac{2 + 36i}{10}$$

Now, $|z| = \frac{1}{10}|2 + 36i|$

$$= \frac{1}{10}\sqrt{(2)^2 + (36)^2} = \frac{1}{10}\left(10\sqrt{13}\right) = \sqrt{13}$$

Also, $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{36/10}{2/10}\right) = \tan^{-1} 18$

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32. Let $z_1 = a + ib$ and $z_2 = c + id$

(i) $\overline{(z_1 + z_2)} = \overline{(a+ib)+(c+id)}$

$= \overline{(a+c)+i(b+d)} = (a+c)-i(b+d)$

$= (a-ib)+(c-id) = (\overline{z_1} + \overline{z_2})$

Hence, $\overline{(z_1 + z_2)} = (\overline{z_1} + \overline{z_2})$

(ii) $\overline{(z_1 - z_2)} = \overline{(a+ib)-(c+id)}$

$= \overline{(a-c)+i(b-d)} = (a-c)-i(b-d)$

$= (a-ib)-(c-id) = \overline{(a+ib)} - \overline{(c+id)} = (\overline{z_1} - \overline{z_2})$

Hence, $\overline{(z_1 - z_2)} = (\overline{z_1} - \overline{z_2})$

33. We have, $\dfrac{1}{(1-2i)} = \dfrac{1}{(1-2i)} \times \dfrac{(1+2i)}{(1+2i)}$

$= \dfrac{(1+2i)}{(1^2-4i^2)} = \dfrac{(1+2i)}{5} = \left(\dfrac{1}{5} + \dfrac{2}{5}i\right)$

Also, $\dfrac{3}{(1+i)} = \dfrac{3}{(1+i)} \times \dfrac{(1-i)}{(1-i)}$

$= \dfrac{3(1-i)}{(1^2-i^2)} = \dfrac{3(1-i)}{2} = \left(\dfrac{3}{2} - \dfrac{3}{2}i\right)$

Now, $\dfrac{(3+4i)}{(2-4i)} = \dfrac{(3+4i)}{(2-4i)} \times \dfrac{(2+4i)}{(2+4i)}$

$= \dfrac{(6-16)+(12+8)i}{(4-16i^2)} = \dfrac{-10+20i}{20} = \left(\dfrac{-1}{2} + i\right)$

$\therefore$ Given expression $= \left\{\left(\dfrac{1}{5} + \dfrac{2}{5}i\right) + \left(\dfrac{3}{2} - \dfrac{3}{2}i\right)\right\}\left(-\dfrac{1}{2} + i\right)$

$= \left\{\left(\dfrac{1}{5} + \dfrac{3}{2}\right) + \left(\dfrac{2}{5} - \dfrac{3}{2}\right)i\right\}\left\{\dfrac{-1+2i}{2}\right\}$

$= \left(\dfrac{17}{10} - \dfrac{11}{10}i\right)\left(\dfrac{-1+2i}{2}\right) = \dfrac{(17-11i)}{10} \times \dfrac{(-1+2i)}{2}$

$= \dfrac{(-17+22)+(34+11)i}{20} = \dfrac{(5+45i)}{20}$

$= \left(\dfrac{5}{20} + \dfrac{45}{20}i\right) = \left(\dfrac{1}{4} + \dfrac{9}{4}i\right)$

34. Let $z = (x + iy)$. Then, $|z| + z = (2 + i)$

$\Rightarrow \quad |x + iy| + (x + iy) = (2 + i)$

$\Rightarrow \quad \left\{\sqrt{x^2 + y^2} + x\right\} + iy = (2 + i)$

$\Rightarrow \quad \sqrt{x^2 + y^2} + x = 2$ and $y = 1$

[Equating real and imaginary parts separately]

$\Rightarrow \quad \sqrt{x^2 + y^2} = (2 - x)$ and $y = 1$

$\Rightarrow \quad \sqrt{x^2 + 1} = (2 - x)$ and $y = 1$ $\qquad [\because y = 1]$

$\Rightarrow \quad (x^2 + 1) = (2 - x)^2$ and $y = 1$

$\Rightarrow \quad (x^2 + 1) = (4 + x^2 - 4x)$ and $y = 1$

$\Rightarrow \quad 4 - 4x = 1$ and $y = 1 \quad \Rightarrow \quad x = \dfrac{3}{4}$ and $y = 1.$

Hence, $z = \left(\dfrac{3}{4} + i\right).$

OR

We have, $a + ib = \dfrac{c+i}{c-i}$

$\Rightarrow \quad a + ib = \dfrac{c+i}{c-i} \times \dfrac{c+i}{c+i} = \dfrac{(c+i)^2}{c^2 - i^2}$

$\Rightarrow \quad a + ib = \dfrac{c^2 + i^2 + 2ic}{c^2 + 1} = \dfrac{c^2 - 1 + 2ic}{c^2 + 1}$

$\Rightarrow \quad a + ib = \dfrac{c^2 - 1}{c^2 + 1} + \dfrac{2c}{c^2 + 1}i$

On comparison, we get

$$a = \frac{c^2 - 1}{c^2 + 1}, b = \frac{2c}{c^2 + 1}$$

Hence, $a^2 + b^2 = \dfrac{c^4 + 1 - 2c^2 + 4c^2}{(c^2 + 1)^2}$

$\Rightarrow \quad a^2 + b^2 = \dfrac{(c^2 + 1)^2}{(c^2 + 1)^2} = 1$

Also, $\dfrac{b}{a} = \dfrac{2c}{c^2 + 1} \cdot \dfrac{c^2 + 1}{c^2 - 1} = \dfrac{2c}{c^2 - 1}$

35. We have, $|z_1| = |z_2| = |z_3| = ... = |z_n| = 1$

$\Rightarrow \quad |z_1|^2 = |z_2|^2 = |z_3|^2 = ... = |z_n|^2 = 1$

$\Rightarrow \quad z_1\overline{z}_1 = 1, z_2\overline{z}_2 = 1, z_3\overline{z}_3 = 1, ..., z_n\overline{z}_n = 1$

$\Rightarrow \quad \dfrac{1}{z_1} = \overline{z}_1, \dfrac{1}{z_2} = \overline{z}_2, \dfrac{1}{z_3} = \overline{z}_3, ..., \dfrac{1}{z_n} = \overline{z}_n$

$\Rightarrow \quad \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} + ... + \dfrac{1}{z_n}\right| = \left|\overline{z}_1 + \overline{z}_2 + \overline{z}_3 + ... + \overline{z}_n\right|$

$= \overline{|z_1 + z_2 + z_3 + ... + z_n|}$

$= |z_1 + z_2 + z_3 + ... + z_n| \qquad [\because \ |\overline{z}| = |z|]$

$\therefore \quad \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} + ... + \dfrac{1}{z_n}\right| = |z_1 + z_2 + z_3 + ... + z_n|$

OR

Let $z = x + iy$

$\Rightarrow \quad z^2 = x^2 - y^2 + 2ixy$ and $|z|^2 = x^2 + y^2$

Given, $2|z|^2 + z^2 - 5 + i\sqrt{3} = 0$

$\Rightarrow \quad 2(x^2 + y^2) + x^2 - y^2 + 2ixy - 5 + i\sqrt{3} = 0$

$\Rightarrow \quad 2x^2 + 2y^2 + x^2 - y^2 - 5 + i(2xy + \sqrt{3}) = 0 + 0i$

Equating real and imaginary parts, we get

$3x^2 + y^2 - 5 = 0$ ...(i)

and $2xy + \sqrt{3} = 0 \Rightarrow 2xy = -\sqrt{3}$

$\Rightarrow \quad y = \dfrac{-\sqrt{3}}{2x}$

Substituting this value of y in (i), we get

$$3x^2 + \frac{3}{4x^2} - 5 = 0$$

$\Rightarrow \quad 12x^4 - 20x^2 + 3 = 0$

$\Rightarrow \quad 12x^4 - 18x^2 - 2x^2 + 3 = 0$

$\Rightarrow \quad 6x^2(2x^2 - 3) - 1(2x^2 - 3) = 0$

$\Rightarrow \quad (6x^2 - 1)(2x^2 - 3) = 0$

$\Rightarrow \quad x = \pm \dfrac{1}{\sqrt{6}}, \ \pm\sqrt{\dfrac{3}{2}}$

When $x = \dfrac{1}{\sqrt{6}}$, then $y = \dfrac{-3\sqrt{2}}{2} = \dfrac{-3}{\sqrt{2}}$

When $x = \dfrac{-1}{\sqrt{6}}$, then $y = \dfrac{3\sqrt{2}}{2} = \dfrac{3}{\sqrt{2}}$

When $x = \sqrt{\dfrac{3}{2}}$, then $y = \dfrac{-1}{\sqrt{2}}$

When $x = -\sqrt{\dfrac{3}{2}}$, then $y = \dfrac{1}{\sqrt{2}}$

$\therefore$ Possible values of z are

$$\frac{1}{\sqrt{6}} - i\frac{3}{\sqrt{2}}, \ \frac{-1}{\sqrt{6}} + i\frac{3}{\sqrt{2}}, -\sqrt{\frac{3}{2}} + i\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} - i\frac{1}{\sqrt{2}}$$

36. We know that if a polynomial $f(x)$ is divided by $x - \alpha$, then the remainder is $f(\alpha)$.

Here, $1 + i$ and $1 - i$ are the remainders when $f(z)$ is divided by $z + 2i$ and $z - 2i$.

$\therefore \quad f(-2i) = 1 + i$ and $f(2i) = 1 - i$.

Let $lz + m$ be the remainder when $f(z)$ is divided by $z^2 + 4$ and $q(z)$ be the quotient in that case.

$\therefore \quad f(z) = (z^2 + 4)q(z) + lz + m$

$\Rightarrow \quad f(-2i) = 0 + l(-2i) + m$

$\Rightarrow \quad 1 + i = -2li + m$ ...(i)

Also, $f(2i) = 0 + l(2i) + m$

$\Rightarrow \quad 1 - i = 2li + m$ ...(ii)

Adding (i) and (ii), we get $2m = 2 \Rightarrow m = 1$,

$\therefore \quad l = -\dfrac{1}{2}$

$\therefore$ Required remainder $= lz + m = -\dfrac{1}{2}z + 1$

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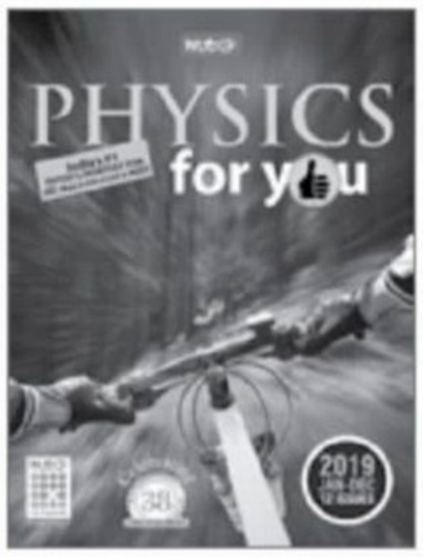

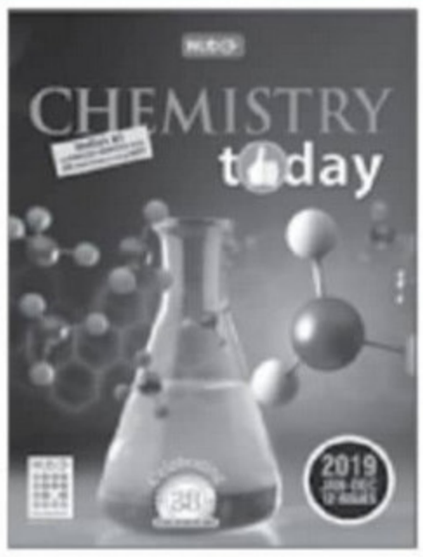

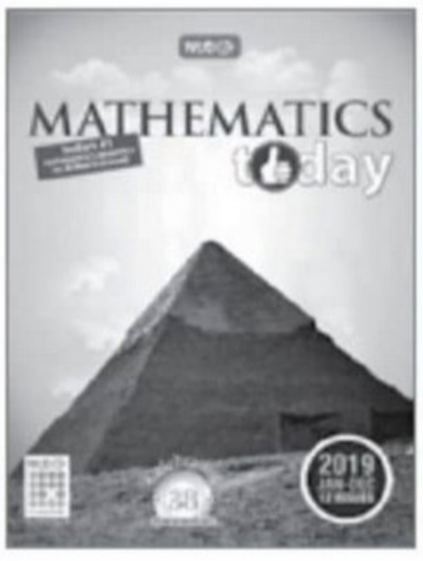

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MONTHLY TEST DRIVE

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80

Series 3 : Trigonometric Functions

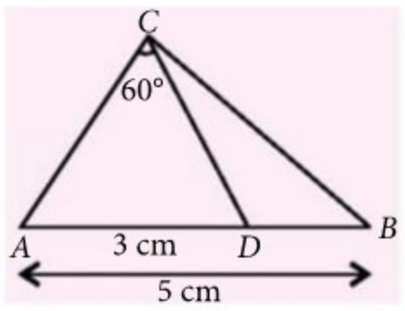
Time Taken : 60 Min.

Only One Option Correct Type

1. In a ΔABC if $(\sqrt{3}-1)a = 2b$, $A = 3B$, then C is
(a) 60° (b) 120° (c) 30° (d) 45°

2. The maximum value of $1 + \sin\left(\frac{\pi}{4}+\theta\right) + 2\cos\left(\frac{\pi}{4}-\theta\right)$ for real values of θ is
(a) 3 (b) 5
(c) 4 (d) none of these

3. In the figure, ABC is a triangle in which $\angle C = 90^\circ$ and $AB = 5$ cm. D is a point on AB such that $AD = 3$ cm and $\angle ACD = 60^\circ$. Then the length of AC is



(a) $5\sqrt{\frac{3}{7}}$ cm (b) $\sqrt{\frac{7}{3}}$ cm
(c) $\frac{\sqrt{3}}{7}$ cm (d) none of these

4. If $\sin 6\theta + \sin 4\theta + \sin 2\theta = 0$, then the general value of θ is
(a) $\frac{n\pi}{4}, n\pi \pm \frac{\pi}{3}$ (b) $\frac{n\pi}{4}, n\pi \pm \frac{\pi}{6}$
(c) $\frac{n\pi}{4}, 2n\pi \pm \frac{\pi}{3}$ (d) $\frac{n\pi}{4}, 2n\pi \pm \frac{\pi}{6}$

5. If $\tan\theta \cdot \tan(120^\circ - \theta) \cdot \tan(120^\circ + \theta) = \frac{1}{\sqrt{3}}$, then $\theta =$
(a) $\frac{n\pi}{3} + \frac{\pi}{18}, n \in Z$ (b) $\frac{n\pi}{3} + \frac{\pi}{12}, n \in Z$
(c) $\frac{n\pi}{12} + \frac{\pi}{12}, n \in Z$ (d) $\frac{n\pi}{3} + \frac{\pi}{6}, n \in Z$

6. The value of $\cot 16^\circ \times \cot 44^\circ + \cot 44^\circ \times \cot 76^\circ - \cot 76^\circ \times \cot 16^\circ$ is
(a) 4 (b) 1 (c) 3 (d) 0

One or More Than One Option(s) Correct Type

7. If $\sin A = a\cos B$ and $\cos A = b\sin B$, then $(a^2 - 1)\tan^2 A + (1 - b^2)\tan^2 B$ is equal to
(a) $\frac{a^2 - b^2}{b^2}$ (b) $\frac{a^2 - b^2}{a^2}$
(c) $\frac{a^2 + b^2}{b^2}$ (d) $\frac{a^2 + b^2}{a^2}$

8. If $A + B = \frac{\pi}{3}$ and $\cos A + \cos B = 1$, then which of the following is/are true?
(a) $\cos(A - B) = \frac{1}{3}$ (b) $|\cos A - \cos B| = \sqrt{\frac{2}{3}}$
(c) $\cos(A - B) = -\frac{1}{3}$ (d) $|\cos A - \cos B| = \frac{1}{2\sqrt{3}}$

9. If $0 < \alpha, \beta < \pi$ and $\cos\alpha + \cos\beta - \cos(\alpha + \beta) = 3/2$, then
(a) $\alpha = \pi/3$ (b) $\beta = \pi/3$
(c) $\alpha \neq \beta$ (d) $\alpha + \beta = \pi/3$

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10. If $\sin(\alpha+\beta)=1$ and $\sin(\alpha-\beta)=1/2$ where $\alpha,\beta\in[0,\pi/2]$ then

(a) $\tan(\alpha+2\beta)=-\sqrt{3}$

(b) $\tan(2\alpha+\beta)=-1/\sqrt{3}$

(c) $\tan(\alpha+2\beta)=\sqrt{3}$

(d) $\tan(2\alpha+\beta)=1/\sqrt{3}$

11. If $\cos^3 x\sin 2x=\sum_{m=1}^{n} a_m \sin mx$ is an identity in x, then which one of the following is correct?

(a) $a_2=0, a_3=\frac{3}{8}$ (b) $a_1=\frac{1}{2}, n=6$

(c) $a_1=\frac{1}{4}, n=5$ (d) $\sum_{m=1}^{n} a_m=\frac{3}{4}$

12. If sides of triangle ABC are a, b and c such that $2b=a+c$, then

(a) $\frac{b}{c}>\frac{2}{3}$ (b) $\frac{b}{c}>\frac{1}{3}$

(c) $\frac{b}{c}<2$ (d) $\frac{b}{c}<\frac{1}{2}$

13. If $4\sin^4 x+\cos^4 x=1$, then x is equal to

(a) $n\pi, n\in Z$ (b) $n\pi\pm\sin^{-1}\sqrt{\frac{2}{5}}, n\in Z$

(c) $\frac{2n\pi}{3}, n\in Z$ (d) $2n\pi\pm\frac{\pi}{4}, n\in Z$

Comprehension Type

Given that $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$ are the roots of the equation $8x^2-4x^2-4x+1=0$.

14. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{7}}{8}$

15. The value of $\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14}$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{7}}{8}$

Matrix Match Type

16. Match the following:

	Column-I		Column-II
P.	Max $\{5\sin\theta+3\sin(\theta-\alpha), \theta\in R\}=7$, then the set of possible values of α is	1.	$2n\pi+3\pi/4, n\in Z$
Q.	If $x\neq\frac{(2n+1)\pi}{2}$ and $(\cos x)^{(\sin^2 x-3\sin x+2)}=1$, then $x=$	2.	$2n\pi\pm\frac{\pi}{3}, n\in Z$
R.	If $\sqrt{(\sin x)}+2^{1/4}\cos x=0$, then $x=$	3.	$2n\pi+\cos^{-1}(1/3), n\in Z$
S.	If $\log_5\tan x=(\log_5 4)(\log_4(3\sin x))$, then $x=$	4.	$2n\pi, n\in Z$

	P	Q	R	S
(a)	1	3	4	2
(b)	2	4	1	3
(c)	4	2	3	1
(d)	4	1	2	3

Integer Answer Type

17. If $\cos 5\theta=a\cos^5\theta+b\cos^3\theta+c\cos\theta$, then $c=$

18. If $3\sin x+4\cos x=5$, then the value of $90\tan^2(x/2)-60\tan(x/2)+16$ is equal to

19. If $k\left[\tan^6\frac{\pi}{9}-33\tan^4\frac{\pi}{9}+27\tan^2\frac{\pi}{9}\right]=9$, then the value of k is

20. If $A+B+C=\pi$ and $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=\lambda\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$, then the value of λ must be

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SELF CHECK

No. of questions attempted .....
No. of questions correct .....
Marks scored in percentage .....

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT MAP

BINOMIAL THEOREM

Class XI

Expansions

positive integral index 'n', $n \in N$

$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + ... + {}^nC_na^n$

$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n, \left|\frac{x}{a}\right|<1$

$(1+x)^n = \sum_{r=0}^{n} {}^nC_rx^r$

$(1-x)^n = \sum_{r=0}^{n}(-1)^r\,{}^nC_rx^r$

integral index '$-n$', $n \in N$

$(1+x)^{-n} = \sum_{r=0}^{n}(-1)^r\,{}^nC_rx^r, (1-x)^{-n} = \sum_{r=0}^{n}{}^nC_rx^r$

General Term

- $T_{r+1} = {}^nC_r\,x^{n-r}\,a^r$
- T_{r+1} from the end = T_{n-r+1} from the beginning

Middle Terms

- If n is even, $T_{\frac{n}{2}+1}$ is the middle term.
- If n is odd, $T_{\frac{n+1}{2}}, T_{\frac{n+3}{2}}$ are two middle terms.

Multinomial Theorem

$(x_1+x_2+x_3+...+x_r)^n = \sum \frac{n!}{p_1!\,p_2!\,...\,p_r!}x_1^{p_1}x_2^{p_2}...x_r^{p_r}$

Number of Terms in the Expansion

Number of terms in the expansion of $(x+y+z)^n$

$= x^n + {}^nC_1x^{n-1}(y+z) + {}^nC_2x^{n-2}(y+z)^2 + ... + {}^nC_n(y+z)^n$

$= 1 + 2 + 3 + ... + (n+1) = \sum_{r=1}^{n+1} r = \frac{1}{2}(n+1)(n+2)$

The number of terms in $(x_1 + x_2 + x_3 + ... + x_k)^n$ is equal to the number of non-negative integral solutions of the equation $r_1 + r_2 + ... + r_k = n$

The number of such solutions = ${}^{n+k-1}C_{k-1}$.

Binomial Theorem for Any Index

- $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + ... + \frac{n(n-1)...(n-r+1)x^r}{r!}$ $+ ... \infty, |x|<1$
- $T_{r+1} = \frac{n(n-1)(n-2)...(n-r+1)}{r!}x^r$
- T_{r+1} in $(1+x)^{-n} = \frac{(-1)^r\,n(n+1)(n+2)...(n+r-1)}{r!}x^r$
- T_{r+1} in $(1-x)^{-n} = \frac{n(n+1)(n+2)...(n+r-1)}{r!}x^r$

Binomial Coefficients

Greatest Binomial Coefficient is

- ${}^nC_{\frac{n}{2}}$, if n is even
- ${}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$, if n is odd

Some Important Results

- ${}^nC_r = {}^nC_{n-r}$
- ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}\left(\frac{n-1}{r-1}\right){}^{n-2}C_{r-2}$
- ${}^nC_x = {}^nC_y \Rightarrow$ Either $x = y$ or $x + y = n$
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- $\sum_{r=0}^{n}{}^nC_r = \sum_{r=0}^{n}C_r = 2^n$
- $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$
- $\sum_{r=0}^{n} r \cdot C_r = n2^{n-1}$
- $C_0^2 - C_1^2 + C_2^2 - ... + (-1)^nC_n^2 = \begin{cases} 0, n \text{ is odd} \\ (-1)^{\frac{n}{2}}\,{}^nC_{\frac{n}{2}}, n \text{ is even} \end{cases}$
- $C_0 - C_2 + C_4 - C_6 + ... = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$
- $C_1 - C_3 + C_5 - C_7 + ... = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$
- If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} ... p_r^{\alpha_r}$, where $p_1, p_2, ..., p_r$ are prime numbers and $\alpha_1, \alpha_2, ..., \alpha_r$ are positive integers, then sum of positive divisors of $n = (1 + p_1 + p_1^2 + ... + p_1^{\alpha_1})(1 + p_2 + p_2^2 + ... + p_2^{\alpha_2}) ... (1 + p_r + p_r^2 + ... + p_r^{\alpha_r})$.

APPLICATION OF DERIVATIVES

CONCEP MAP

Class XII

Rate of Change of Quantities

- Let $y = f(x)$ then $\frac{dy}{dx}$ or $f'(x)$ denotes the rate of change of y w.r.t. x and its value at $x = a$ is denoted as $\left[\frac{dy}{dx}\right]_{x=a}$.

Marginal Cost and Marginal Revenue

- Let C be the total cost of producing and marketing x units of a product, then marginal cost (MC), is $MC = \frac{dC}{dx}$.
- The rate of change of total revenue with respect to the quantity sold is the marginal revenue, $MR = \frac{dR}{dx}$.

Errors and Approximations

Let $y = f(x)$, Δx be the small change in x and Δy be the corresponding change in y. Then, $\Delta y = \frac{dy}{dx}(\Delta x)$

These small values Δx and Δy are called differentials.

(i) Absolute Error : Δx (ii) Relative Error : $\frac{\Delta x}{x}$

(iii) Percentage Error : $\left(\frac{\Delta x}{x} \times 100\right)$

Increasing and Decreasing Functions

Increasing Function

Increasing Function	without derivative test	If $x_1 < x_2 \Rightarrow f(x_1) \leq f($ $x_1, x_2 \in (a, b)$
	with derivative test	If $f'(x) \geq 0$ for each $x \in ($
Strictly Increasing Function	without derivative test	If $x_1 < x_2 \Rightarrow f(x_1) < f($ $x_1, x_2 \in (a, b)$
	with derivative test	If $f'(x) > 0$ for each $x \in$

Decreasing Function

Decreasing Function	without derivative test	If $x_1 < x_2 \Rightarrow f(x_1) \geq f($ $x_1, x_2 \in (a, b)$
	with derivative test	If $f'(x) \leq 0$ for each $x \in$
Strictly Decreasing Function	without derivative test	If $x_1 < x_2 \Rightarrow f(x_1) > f($ $x_1, x_2 \in (a, b)$
	with derivative test	If $f'(x) < 0$ for each $x \in$

Maxima And Minima

They exist at critical points only.

Critical Points

1. $f(x)$ doesn't exist
2. $f'(x)$ doesn't exist
3. $f'(x) = 0$

Global Max., Global Min.

Global Max. : Max. value of $f(x)$ in domain.

Global Min. : Least value of $f(x)$ in domain.

Local Maxima And Local Min

Local Maxima : At $x = a$

$f(a - h) < f(a) > f(a + h)$ $(h \to$

Local Minima : At $x = a$

$f(a - h) > f(a) < f(a + h)$ $(h \to$

First derivative test

1. $x = a$ is local max. if $f'(a) = 0$ and $f'(x)$ changes from +ve to –ve.
2. $x = a$ is local min. if $f'(a) = 0$ and $f'(x)$ changes from –ve to +ve.

Second derivative test

1. Find the roots of $f'(x) = 0$
2. Suppose $x = a$ is one of the roots.
3. At those points, if $f''(x) < 0 \Rightarrow f(x)$ is maximum at $x = a$, or if $f''(x) > 0 \Rightarrow f(x)$ is minimum at $x = a$.

Higher order derivative te

1. If $f''(a) = 0$ check $f'''(x)$ at
   If $f'''(a) = 0$, neither max. no
2. $f'''(a) = 0$ repeat process consi
   $f'(x)$ as $g(x)$.

Master the Concepts

Class XI

OF COMPLEX NUMBERS

COMPLEX NUMBERS IN ALGEBRAIC FORM

Every complex number $z = (x, y)$ can be uniquely represented in the form $z = x + yi$
where x, y are real numbers.
The relation $i^2 = -1$ holds.
The real number $x = \text{Re}(z)$ is called the real part of the complex number z and similarly, $y = \text{Im}(z)$ is called the imaginary part of z. Complex numbers of the form iy, $y \in R$, are called purely imaginary number and the complex number i is called the iota.

The following relations are easy to verify :

- $z_1 = z_2$ if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.
- $z \in R$ if and only if $\text{Im}(z) = 0$.
- $z \in C - R$ if and only if $\text{Im}(z) \neq 0$

OPERATIONS WITH COMPLEX NUMBERS

- $z_1 + z_2 = (x_1 + y_1 i) + (x_2 + y_2 i)$
  $= (x_1 + x_2) + (y_1 + y_2)i \in C$
- $z_1 - z_2 = (x_1 + y_1 i) - (x_2 + y_2 i)$
  $= (x_1 - x_2) + (y_1 - y_2)i \in C$
- $z_1 \cdot z_2 = (x_1 + y_1 i)(x_2 + y_2 i)$
  $= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i \in C$
- $\lambda(z_1 + z_2) = \lambda z_1 + \lambda z_2$; $\lambda_1(\lambda_2 z) = (\lambda_1 \lambda_2)z$; and $(\lambda_1 + \lambda_2)z = \lambda_1 z + \lambda_2 z$ for all $z, z_1, z_2 \in C$ and $\lambda, \lambda_1, \lambda_2 \in R$.

POWERS OF THE NUMBER *i*

One can prove by induction that for every positive integer n,

$i^{4n} = 1$; $i^{4n+1} = i$; $i^{4n+2} = -1$; $i^{4n+3} = -i$

Hence $i^n \in \{-1, 1, -i, i\}$ for all integers $n \geq 0$. If n is a negative integer, we have

$$i^n = (i^{-1})^{-n} = \left(\frac{1}{i}\right)^{-n} = (-i)^{-n}.$$

CONJUGATE OF A COMPLEX NUMBER

For a complex number $z = x + yi$ the number $\overline{z} = x - yi$ is called the complex conjugate of z or occasionally the conjugate complex number of z.

Properties

(1) The relation $z = \overline{z}$ holds if and only if $z \in R$.

(2) For every complex number z, the relation $z = \overline{(\overline{z})}$ holds.

(3) For every complex number z, the number $z \cdot \overline{z} \in R$ is a non-negative real number.

(4) $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$ (the conjugate of a sum is the sum of the conjugates).

(5) $\overline{z_1 \cdot z_2} = \overline{z}_1 \cdot \overline{z}_2$ (the conjugate of a product is the product of the conjugates).

(6) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$, $z_2 \neq 0$ (the conjugate of a quotient is the quotient of the conjugates).

(7) For every non-zero complex number z, the relation $\overline{z^{-1}} = (\overline{z})^{-1}$ holds.

(8) The formulas $\text{Re}(z) = \frac{z + \overline{z}}{2}$ and $\text{Im}(z) = \frac{z - \overline{z}}{2i}$ are valid for all $z \in C$.

(9) Properties (4) and (5) can be easily extended to give

(4′) $\overline{\left(\sum_{k=1}^{n} z_k\right)} = \sum_{k=1}^{n} \overline{z}_k$;

(5′) $\overline{\left(\prod_{k=1}^{n} z_k\right)} = \prod_{k=1}^{n} \overline{z}_k$ for all $z_k \in C$, $k = 1, 2, \ldots, n$.

As a consequence of (5′) and (7), we have

(5″) $\overline{(z^n)} = (\overline{z})^n$ for every integer n and for every $z \in C$. In $n < 0$, then the formula holds for every $z \neq 0$.

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THE MODULUS OF A COMPLEX NUMBER

The number $|z| = \sqrt{x^2 + y^2}$ is called the modulus or the absolute value of the complex number $z = x + yi$.

Properties

(1) $-|z| \le \text{Re}(z) \le |z|$ and $-|z| \le \text{Im}(z) \le |z|$.

(2) $|z| \ge 0$ for all $z \in C$. Moreover, we have $|z| = 0$ if and only if $z = 0$.

(3) $|z| = |-z| = |\overline{z}|$

(4) $z \cdot \overline{z} = |z|^2$

(5) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ (the modulus of a product is the product of the moduli).

(6) $|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|$.

(7) $|z^{-1}| = |z|^{-1}$, $z \ne 0$.

(8) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \ne 0$ (the modulus of a quotient is the quotient of the moduli).

(9) $|z_1| - |z_2| \le |z_1 - z_2| \le |z_1| + |z_2|$.

Remarks

(i) The inequality $|z_1 + z_2| \le |z_1 + z_2|$ becomes an equality if and only if $\text{Re}(z_1 \overline{z}_2) = |z_1||z_2|$. This is equivalent to $z_1 = tz_2$, where t is a non-negative real number.

(ii) Properties (5) and (6) can be easily extended to give

(5′) $\left|\prod_{k=1}^{n} z_k\right| = \prod_{k=1}^{n} |z_k|$;

(6′) $\left|\sum_{k=1}^{n} z_k\right| \le \sum_{k=1}^{n} |z_k|$ for all $z_k \in C$, $k = 1, \ldots, n$.

As a consequence of (5′) and (7), we have

(5″) $|z^n| = |z|^n$ for every integer n and complex number z, provided that $z \ne 0$ for $n < 0$.

GEOMETRIC INTERPRETATION OF A COMPLEX NUMBER

Any complex number $z = x + iy$ represents the point $M(x, y)$ in the x-y plane which is referred to as complex plane or argand plane hereafter. The x-axis is called the real axis and the y-axis is called the imaginary axis.

Let $z = x + iy$ represent the point $M(x, y)$ in the complex plane then the conjugate $\overline{z} = x - iy$ of the complex number z is the image of the point $M(x, y)$ in the x-axis.

Similarly, the additive inverse $-z$ of the complex number z is the image of the point $M(x, y)$ about origin.

POLAR AND EULER REPRESENTATION OF A COMPLEX NUMBER

For a complex number $z = x + iy$, we can write the polar representation as $z = r(\cos\theta + i \sin\theta)$, where r is called the modulus of complex number z and θ is the principal (reduced) argument of complex number z denoted by $\arg z$. Also, $-\pi < \arg z \le \pi$. To calculate the principal argument of any complex number we first calculate $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$, then α, $\pi - \alpha$, $\alpha - \pi$, $-\alpha$ becomes the principal argument of the complex number according as complex number lies in Ist, IInd, IIIrd or IVth quadrant. For a complex number on positive y-axis, principal argument is taken as $\frac{\pi}{2}$, whereas for a complex number on negative y-axis it is taken as $-\frac{\pi}{2}$.

In Euler form, any complex number is represented as $z = re^{i\theta} = |z|e^{i(\arg z)}$.

Operations with complex numbers in polar representation

If $z_1 = r_1(\cos\theta_1 + i \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i \sin\theta_2)$, then

$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$ and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)) \qquad \ldots(1)$$

Remarks

(i) We find again that $|z_1 z_2| = |z_1| \cdot |z_2|$.

(ii) We have $\arg(z_1 z_2) = \arg z_1 + \arg z_2 - 2k\pi$, where

$$k = \begin{cases} 0 \text{ or } -1 & \text{for } \arg z_1 + \arg z_2 < 2\pi \\ 1, & \text{for } \arg z_1 + \arg z_2 \ge 2\pi \end{cases}$$

Also, we can write $\arg(z_1 z_2) = \{\arg z_1 + \arg z_2 + 2k\pi : k \in Z\}$.

(iii) Formula (1) can be extended to $n \ge 2$ complex numbers. If $z_k = r_k(\cos t_k + i \sin t_k)$, $k = 1, \ldots, n$, then $z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n(\cos(t_1 + t_2 + \ldots + t_n) + i \sin(t_1 + t_2 + \ldots + t_n))$.

The proof by induction is immediate. This formula can be written as

$$\prod_{k=1}^{n} z_k = \prod_{k=1}^{n} r_k \left(\cos \sum_{k=1}^{n} t_k + i \sin \sum_{k=1}^{n} t_k\right).$$

(iv) We have again $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$.

(v) We can write

$$\arg\left(\frac{z_1}{z_2}\right) = \{\arg z_1 - \arg z_2 + 2k\pi : k \in Z\}.$$

THE POWER OF A COMPLEX NUMBER

(De Moivre's Theorem) For $z = r(\cos t + i \sin t)$ and $n \in N$, we have $z^n = r^n(\cos nt + i \sin nt)$.

Remarks

(i) We find again that $|z^n| = |z|^n$.

(ii) If $r = 1$, then $(\cos t + i \sin t)^n = \cos nt + i \sin nt$.

(iii) We can write $\arg z^n = \{n \arg z + 2k\pi : k \in Z\}$.

(iv) De Moivre's theorem also holds for negative integer exponents n, *i.e.*, we have
$z^n = r^n (\cos nt + i \sin nt)$, where $n \in Z$.

(v) $\frac{1}{z} = z^{-1} = \frac{1}{r}(\cos(-t) + i \sin(-t))$.

n^{th} ROOTS OF A COMPLEX NUMBER

Let $z_0 = r(\cos\theta + i \sin\theta)$, be a complex number written in polar form, then the complex number z_0 has n distinct n^{th} roots, given by the formula :

$$z_k = r^{1/n}\left(\cos\frac{\theta+2k\pi}{n} + i \sin\frac{\theta+2k\pi}{n}\right),$$

$k = 0, 1, 2, \ldots, (n-1)$.

THE n^{th} ROOTS OF UNITY

The roots of the equation $z^n - 1 = 0$ are called the n^{th} roots of unity. Since, $1 = \cos 0 + i \sin 0$, from the formula for the n^{th} roots of a complex number, we derive that the n^{th} roots of unity are

$$z_k\left(\cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}\right),\ k = 0, 1, 2, \ldots, (n-1).$$

Let $\alpha = \left(\cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}\right)$, then the n^{th} roots of unity are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$

We observe that $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha} = 0$

i.e., sum of the n^{th} roots of unity is zero. The n^{th} roots of unity are the vertices of a regular polygon with n sides inscribed in the unit circle with one of the vertices at 1.

THE CUBE ROOTS OF UNITY

For $n = 3$, the cube roots of unity *i.e.*, the roots of equation $z^3 - 1 = 0$, are given by

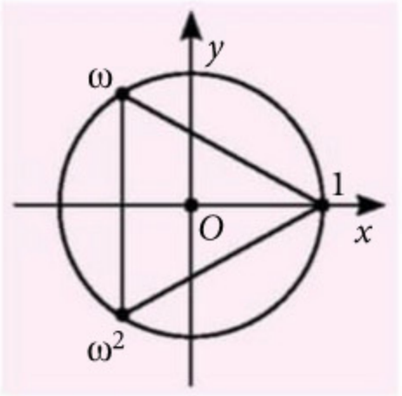
$$z_k = \left(\cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}\right),\ k = 0, 1, 2.$$

These are denoted by $1, \omega, \omega^2$, where

$$\omega = \left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ and}$$

$$\omega^2 = \left(\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

They form an equilateral triangle inscribed in the unit circle with one of the vertices at 1.

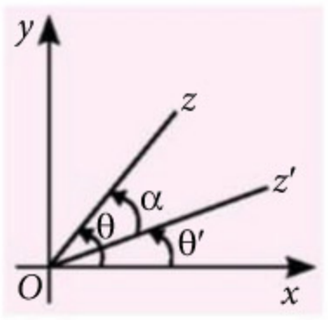


Properties of the cube roots of unity

- $\omega^3 = 1$
- $1 + \omega + \omega^2 = 0$
- $\omega^{3n} = 1, \omega^{3n+1} = \omega, \omega^{3n+2} = \omega^2$
- ω and ω^2 are conjugate of each other
- $|\omega| = 1$

CONCEPT OF ROTATION

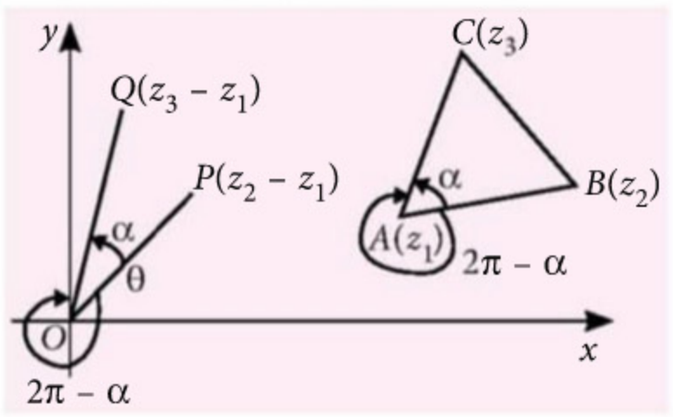
If z and z' are two complex numbers then argument of $\frac{z}{z'}$ is the angle through which OZ' must be turned in order that it may lie along OZ.



$$\frac{z}{z'} = \frac{|z|e^{i\theta}}{|z'|e^{i\theta'}} = \frac{|z|}{|z'|}e^{i(\theta-\theta')} = \frac{|z|}{|z'|}e^{i\alpha}$$

In general, let z_1, z_2, z_3, be the three vertices of a triangle ABC described in the counter-clock wise sense. Draw OP and OQ parallel and equal to AB and AC respectively. Then the point P is $z_2 - z_1$ and Q is $z_3 - z_1$ and $\frac{z_3 - z_1}{z_2 - z_1} = \frac{OQ}{OP}(\cos\alpha + i \sin\alpha)$

$$= \frac{CA}{BA} \cdot e^{i\alpha} = \frac{|z_3 - z_1| \cdot e^{i\alpha}}{|z_2 - z_1|}$$



ANGLE BETWEEN TWO LINES

Consider four distinct points $M_i(z_i)$, $i \in \{1, 2, 3, 4\}$. The measure of the angle determined by the lines M_1M_3 and M_2M_4 equals

$$\arg\left(\frac{z_3 - z_1}{z_4 - z_2}\right) \text{ or } \arg\left(\frac{z_4 - z_2}{z_3 - z_1}\right).$$

PROBLEMS

Single Option Correct Type

1. For all complex numbers z lying on the unit circle centred at origin, if the expression $|z + 1| + |z^2 - z + 1| = k$, then the range of k is

(a) $[\sqrt{3}, 3]$ (b) $\left[3, \frac{13}{4}\right]$

(c) $\left[\sqrt{3}, \frac{13}{4}\right]$ (d) $\left[\frac{1}{4}, \sqrt{3}\right]$

2. The number of complex numbers 'z' lying on the circle $|z| = 1$ such that $\left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = 1$ is

(a) 4 (b) 5 (c) 7 (d) 8

3. Let $Z \in C$ be such that $\left|z^3 + \frac{1}{z^3}\right| \le 2$. Then, the maximum value of $\left|z + \frac{1}{z}\right|$ will be

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{3}{2}$ (d) 2

4. Let $z_1, z_2 \in C$ be complex numbers such that $|z_1 + z_2| = \sqrt{3}$ and z_1 and z_2 lie on $|z| = 1$, then value of $|z_1 - z_2|$ is

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $\frac{4}{\sqrt{3}}$

5. Let z_1, z_2, z_3 be complex numbers with $|z_1| = |z_2| = |z_3| = R > 0$. If maximum value of $|z_1 - z_2| \cdot |z_2 - z_3| + |z_3 - z_1| \cdot |z_1 - z_2| + |z_2 - z_3| \cdot |z_3 - z_1|$ is λR^2, then λ is equal to

(a) 3 (b) 6 (c) 9 (d) 12

6. Let z_1, z_2, z_3 be distinct complex numbers such that $|z_1| = |z_2| = |z_3| > 0$. If $z_1 + z_2z_3$, $z_2 + z_3z_1$ and $z_3 + z_1z_2$ are real numbers, then $z_1z_2z_3$ is equal to

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{i}{2}$ (d) i

7. Let $(1 - \sqrt{3}i)^n = x_n + iy_n$ where x_n, y_n are real for $n = 1, 2, 3, \ldots\ldots$, then the value of $\left(\frac{x_{100}y_{99} - x_{99}y_{100}}{x_{100}x_{99} + y_{100}y_{99}}\right)$ is

(a) 2 (b) $\frac{\sqrt{3}}{2}$ (c) $\sqrt{3}$ (d) $\frac{2}{\sqrt{3}}$

8. Let z_1, z_2, z_3, z_4 be the roots of $\left(\frac{z - i}{2z - i}\right)^4 = 1$, then the value of $(1 + z_1^2)(1 + z_2^2)(1 + z_3^2)(1 + z_4^2)$ is

(a) $\frac{13}{45}$ (b) $\frac{23}{45}$ (c) $\frac{13}{55}$ (d) $\frac{23}{55}$

9. The number of ordered pairs (a, b) of real numbers such that $(a + ib)^{2002} = (a - ib)$ is

(a) 2003 (b) 2004 (c) 2005 (d) 2006

10. Two regular polygons are inscribed in the same circle. The first polygon has 1982 sides and the second has 2973 sides. If the polygons have any common vertices, how many such vertices will there be?

(a) 991 (b) 992 (c) 993 (d) 994

11. Let z_1, z_2, z_3 be the co-ordinates of the vertices of a triangle such that $\frac{z_2 - z_1}{z_3 - z_1} = \frac{z_3 - z_2}{z_1 - z_2}$, then the value of $\frac{1}{z - z_1} + \frac{1}{z - z_2} + \frac{1}{z - z_3}$, where $z = \frac{z_1 + z_2 + z_3}{3}$ is equal to

(a) 0 (b) 1

(c) 3 (d) none of these

One or More than One Option(s) Correct Type

12. The equation $z^2 + \alpha z + \beta = 0$, where α, β are complex numbers has a

(a) real root if $(\bar{\alpha} - \alpha)(\alpha\bar{\beta} - \bar{\alpha}\beta) = (\beta - \bar{\beta})^2$

(b) real root if $(\bar{\alpha} - \alpha)(\alpha\bar{\beta} + \bar{\alpha}\beta) = (\beta - \bar{\beta})^2$

(c) a purely imaginary root if $(\alpha + \bar{\alpha})(\alpha\bar{\beta} + \bar{\alpha}\beta) + (\beta - \bar{\beta})^2 = 0$

(d) a purely imaginary root if $(\alpha + \bar{\alpha})(\alpha\bar{\beta} - \bar{\alpha}\beta) + (\beta - \bar{\beta})^2 = 0$

13. Let z_1 be on $|z - 3 + 2i| = 4$ and z_2 be on $|z + 3| = 1$ such that $|z_1 - z_2|$ is minimum, then

(a) $z_2 = \left(-3 - \frac{3}{\sqrt{10}}\right) + \frac{i}{\sqrt{10}}$

(b) $z_2 = \left(-3 + \frac{3}{\sqrt{10}}\right) - \frac{i}{\sqrt{10}}$

(c) $z_1 = \left(3 + \frac{12}{\sqrt{10}}\right) + i\left(-2 - \frac{4}{\sqrt{10}}\right)$

(d) $z_1 = \left(3 - \frac{12}{\sqrt{10}}\right) + i\left(-2 + \frac{4}{\sqrt{10}}\right)$

14. Let z_1, z_2, z_3, z_4 be four distinct points in complex plane such that $2|z_4 - z_1| = |z_4 - z_2| + |z_4 - z_3|$, $2|z_4 - z_2| = |z_4 - z_1| + |z_4 - z_3|$, $2|z_4 - z_3| = |z_4 - z_1| + |z_4 - z_2|$ and $\dfrac{z_4 - z_1}{z_3 - z_2}$ is purely imaginary, then

(a) $\dfrac{z_4 - z_2}{z_3 - z_1}$ is purely real

(b) $\dfrac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary

(c) $3z_4 = z_1 + z_2 + z_3$ (d) $|z_4 - z_1| = |z_4 - z_2|$

15. Let two distinct points $P(z_1)$ and $Q(z_2)$ lie on the curve $C: z + \bar{z} = 2|z - 1|$ such that $\arg(z_1 - z_2) = \dfrac{\pi}{4}$, then

(a) P, Q lie on an ellipse

(b) $\text{Im}(z_1 + z_2) = 2$

(c) P, Q lie on a parabola

(d) Point of intersection of tangents to 'C' at P and Q lie on curve $\text{Im}(z) = 1$.

16. The centre of circle circumscribing square $ABCD$ is z_0. Let A is z_1, then centroid of ΔABC can be

(a) $z_0\left(1 - \dfrac{i}{3}\right) - \dfrac{i}{3}z_1$ (b) $z_0\left(1 - \dfrac{i}{3}\right) + \dfrac{i}{3}z_1$

(c) $z_0\left(1 + \dfrac{i}{3}\right) - \dfrac{i}{3}z_1$ (d) $z_0\left(1 + \dfrac{i}{3}\right) + \dfrac{i}{3}z_1$

Comprehension Type

Paragraph for Question No. 17 to 19

On sides AB and BC of ΔABC, squares are drawn with centres D and E respectively such that C and D are on same side of AB and A and E are on opposite sides of BC. If A, B, C are given by complex numbers $1, \omega, \omega^2$ respectively, where $\omega = \dfrac{-1 + \sqrt{3}i}{2}$, then

17. Angle between AC and DE is

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{2}$

18. The length of DE is

(a) $\sqrt{\dfrac{3}{2}}$ (b) $\dfrac{\sqrt{3}}{2}$ (c) $\sqrt{3}$ (d) $\sqrt{6}$

19. The length of AE is

(a) $\dfrac{3 + \sqrt{3}}{4}$ (b) $\dfrac{3 - \sqrt{3}}{2}$ (c) $\dfrac{3 - \sqrt{3}}{4}$ (d) $\dfrac{3 + \sqrt{3}}{2}$

Paragraph for Question No. 20 to 22

Let the point $P(z)$ lies on the curve $C: \left|z + \dfrac{1}{z}\right| = 2$ in the argand plane. Let 'O' denotes origin. Then

20. The largest value of OP is

(a) $\sqrt{5}$ (b) $\sqrt{2}$ (c) $1 + \sqrt{5}$ (d) $1 + \sqrt{2}$

21. If $\arg(z) = \dfrac{\pi}{6}$, then sum of all possible values of $|z|$ is

(a) $\sqrt{5}$ (b) $\sqrt{2}$ (c) 2 (d) $\sqrt{3}$

22. C in argand plane represents

(a) Two concentric circles

(b) Two orthogonal circles

(c) Two non-intersecting circles

(d) Two externally touching circles

Numerical Answer Type

23. Let z_1, z_2, z_3 be the vertices of ΔABC, taken in anti-clockwise direction and z_0 be its circumcentre, then absolute value of $\left(\dfrac{z_0 - z_1}{z_0 - z_2}\right)\dfrac{\sin 2A}{\sin 2B} + \left(\dfrac{z_0 - z_3}{z_0 - z_2}\right)\dfrac{\sin 2C}{\sin 2B}$ is _____ .

24. Let R be the region formed by the set of points $P(x, y)$ in complex plane such that $|z| \le \sqrt{10}$, where $\text{Re}(z) = \sqrt{x^2 + 4}$, $\text{Im}(z) = \sqrt{y - 4}$. If the area of region R is equal to $\sqrt{N}$, then N is equal to _____ .

25. Let $\cos\theta_1 + 2\cos\theta_2 + 3\cos\theta_3 = 0 = \sin\theta_1 + 2\sin\theta_2 + 3\sin\theta_3$, then $\sin 3\theta_1 + 8\sin 3\theta_2 + 27\sin 3\theta_3 = \lambda \sin(\theta_1 + \theta_2 + \theta_3)$ if λ is equal to _____ .

26. If $z + \dfrac{1}{z} = \sqrt{3}$, then $\displaystyle\sum_{r=1}^{5}\left(z^r + \frac{1}{z^r}\right)^2$ is equal to ___.

27. Let A be the area of region in argand plane that consists of all points z such that both $\dfrac{z}{40}$ and $\dfrac{40}{\bar{z}}$ have real and imaginary parts between 0 and 1 inclusive. If $[\cdot]$ denotes greatest integer function, then $[A]$ is equal to _____ .

SOLUTIONS

1. (c) : Let $t = |1 + z| \in [0, 2]$. We have

$t^2 = (1 + z)(1 + \bar{z}) = 2 + 2\text{Re}(z)$, so $\text{Re}(z) = \dfrac{t^2 - 2}{2}$.

$|1 - z + z^2|^2 = (1 - z + z^2)(1 - \bar{z} + \bar{z}^2)$

$= (1 - z + z^2)\left(1 - \dfrac{1}{z} + \dfrac{1}{z^2}\right)$

$= \left(\dfrac{z^2 - z + 1}{z}\right)^2 = \left(z + \dfrac{1}{z} - 1\right)^2 = (z + \bar{z} - 1)^2$

$= [2\text{Re}(z) - 1]^2 = (t^2 - 3)^2$,

The problem reduces to finding maximum and minimum values of

$$f(t) = t + |t^2 - 3| = \begin{cases} -t^2 + t + 3, & \text{if } t \in [0, \sqrt{3}], \\ t^2 + t - 3, & \text{if } t \in [\sqrt{3}, 2], \end{cases}$$

$$\Rightarrow \ f(t)_{\text{min.}} = f\left(\sqrt{3}\right) = \sqrt{3}$$

$$\text{and } \ f(t)_{\text{max.}} = f\left(\frac{1}{2}\right) = \frac{13}{4}$$

2. (d) : Let $z = \cos x + i \sin x$, $x \in [0, 2\pi)$. Then

$$1 = \left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = \frac{|z^2 + \bar{z}^2|}{|z|^2}$$

$$= |\cos 2x + i \sin 2x + \cos 2x - i \sin 2x| = 2|\cos 2x|$$

$$\Rightarrow \ \cos 2x = \frac{1}{2} \text{ or } \cos 2x = -\frac{1}{2}$$

$$\Rightarrow \ x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \text{ or } x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

respectively.

Hence, 8 solutions possible.

3. (d) : $\left(z + \frac{1}{z}\right)^3 = z^3 + \frac{1}{z^3} + 3\left(z + \frac{1}{z}\right)$

$$\Rightarrow \ \left|z + \frac{1}{z}\right|^3 \le 2 + 3\left|z + \frac{1}{z}\right| \ \text{ or } \ a^3 - 3a - 2 \le 0$$

Since, $a = \left|z + \frac{1}{z}\right|$, $a \ge 0$, and

$a^3 - 3a - 2 = (a - 2)(a^2 + 2a + 1) = (a - 2)(a + 1)^2$

$\Rightarrow \ a \le 2$

4. (c) : From $|z_1 + z_2| = \sqrt{3}$ it follows that $|z_1 + z_2|^2 = 3$

i.e., $(z_1 + z_2)(\overline{z_1 + z_2}) = 3$.

$\Rightarrow \ |z_1|^2 + (z_1\bar{z}_2 + \bar{z}_1 z_2) + |z_2|^2 = 3$,

that is, $z_1\bar{z}_2 + \bar{z}_1 z_2 = 1$.

On the other hand, we have

$|z_1 - z_2|^2 = |z_1|^2 - (z_1\bar{z}_2 + \bar{z}_1 z_2) + |z_2|^2 = 2 - 1 = 1$, and hence $|z_1 - z_2| = 1$.

5. (c) : Let $\alpha = |z_2 - z_3|$, $\beta = |z_3 - z_1|$, $\gamma = |z_1 - z_2|$.

Since, $\alpha\beta + \beta\gamma + \gamma\alpha \le \alpha^2 + \beta^2 + \gamma^2$

and $\alpha^2 + \beta^2 + \gamma^2 = 3(|z_1|^2 + |z_2|^2 + |z_3|^2) - |z_1 + z_2 + z_3|^2$

$\le 3(|z_1|^2 + |z_2|^2 + |z_3|^2) = 9R^2$

$\Rightarrow \ \alpha\beta + \beta\gamma + \gamma\alpha \le 9R^2 \ \Rightarrow \ \lambda = 9$

6. (b) : Observe that $z_1 \cdot \bar{z}_1 = z_2 \cdot \bar{z}_2 = z_3 \cdot \bar{z}_3 = r^2$ (say) and $z_1 z_2 + z_3 \in R$ if and only if $z_1 z_2 + z_3 = \bar{z}_1 \cdot \bar{z}_2 + \bar{z}_3$

Then

$$\frac{r^2}{z_1 z_2 z_3} = \frac{z_1 z_2 + z_3}{z_1 z_2 + r^2 z_3} = \frac{z_1 z_3 + z_2}{z_1 z_3 + r^2 z_2} = \frac{z_2 z_3 + z_1}{z_2 z_3 + r^2 z_1}$$

$$= \frac{(z_1 - 1)(z_2 - z_3)}{(z_2 - z_3)(z_1 - r^2)} = \frac{z_1 - 1}{z_1 - r^2} = \frac{z_2 - 1}{z_2 - r^2}$$

$$= \frac{z_3 - 1}{z_3 - r^2} = \frac{z_1 - z_2}{z_1 - z_2} = 1.$$

Hence, $z_1 z_2 z_3 = r^2$, and consequently, $r^3 = r^2$. Therefore, $r = 1$ and $z_1 z_2 z_3 = 1$, as desired.

7. (c) : Let $R(x) = \cos x + i \sin x$. We have

$$(1 - \sqrt{3}i)^n = \left(2R\left(-\frac{\pi}{3}\right)\right)^n = 2^n R\left(-\frac{n\pi}{3}\right),$$

and thus by de Moivre's theorem, we get

$$x_n = 2^n \cos\left(-\frac{n\pi}{3}\right) = 2^n \cos n\theta,$$

$$y_n = 2^n \sin\left(-\frac{n\pi}{3}\right) = -2^n \sin n\theta,$$

where $\theta = \frac{\pi}{3}$.

Substituting these into the given expressions, we obtain

(a) $x_n y_{n-1} - x_{n-1} y_n = 2^{2n-1} \sin(n\theta - (n-1)\theta)$

$= 2^{2n-1} \sin\theta = \sqrt{3} \cdot 4^{n-1}$

(b) $x_n x_{n-1} + y_n y_{n-1} = 2^{2n-1} \cos(n\theta - (n-1)\theta)$

$= 2^{2n-1} \cos\theta = 4^{n-1}$.

Now put $n = 100$ and divide (a) by (b) we get the answer as $\sqrt{3}$.

8. (a) : We have the following equations :

$$1 - \frac{z}{2z - i} = 1; 1 - \frac{z}{2z - i} = -1; \ 1 - \frac{z}{2z - i} = i;$$

$1 - \frac{z}{2z - i} = -i$ which on solving gives $0; \frac{2i}{3}; \frac{3i - 1}{5}; \frac{3i + 1}{5}$

respectively.

$$\Rightarrow \ (1 + z_1^2)(1 + z_2^2)(1 + z_3^2)(1 + z_4^2)$$

$$= (0 + 1)\left(-\frac{4}{9} + 1\right)\left(-\frac{8}{25} - \frac{6i}{25} + 1\right)\left(-\frac{8}{25} + \frac{6i}{25} + 1\right)$$

$$= \left(\frac{5}{9}\right)\left(\frac{13}{25}\right) = \frac{13}{45}$$

9. (b) : Let $z = a + bi$, $\bar{z} = a - bi$, and $|z| = \sqrt{a^2 + b^2}$.

The given relation becomes $z^{2002} = \bar{z}$.

Note that $|z|^{2002} = |z^{2002}| = |\bar{z}| = |z|$, from which it follow that $|z|(|z|^{2001} - 1) = 0$.

Hence $|z| = 0 \ \Rightarrow \ (a, b) = (0, 0)$, or $|z| = 1$.

In the case $|z| = 1$, we have

$z^{2002} = \bar{z}$, which is equivalent to $z^{2003} = \bar{z} \cdot z = |z|^2 = 1$.

Since the equation $z^{2003} = 1$ has 2003 distinct solutions, there are altogether $1 + 2003 = 2004$ ordered pairs that meet the required conditions.

10. (a) : The number of common vertices is given by the number of common roots of $z^{1982} - 1 = 0$ and $z^{2973} - 1 = 0$.

The desired number is HCF (1982, 2973) = 991.

11. (a) : $\dfrac{z_2 - z_1}{z_3 - z_1} = \dfrac{z_3 - z_2}{z_1 - z_2}$

$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

So the triangle is equilateral.

Hence, $\dfrac{1}{z - z_1} + \dfrac{1}{z - z_2} + \dfrac{1}{z - z_3} = 0,$

where $z = \dfrac{z_1 + z_2 + z_3}{3}$

12. (a, c) : For real roots, consider

$z^2 + \alpha z + \beta = 0$ ...(i)

$\bar{z}^2 + \overline{\alpha z} + \bar{\beta} = 0$

$\Rightarrow \quad z^2 + \bar{\alpha} z + \bar{\beta} = 0$ ...(ii)

(i) – (ii) $\Rightarrow \quad z = \dfrac{\bar{\beta} - \beta}{\alpha - \bar{\alpha}}$

Put in (i) we get $(\beta - \bar{\beta})^2 = (\bar{\alpha} - \alpha)(\alpha\bar{\beta} - \bar{\alpha}\beta)$

For purely imaginary roots, consider

$z^2 + \alpha z + \beta = 0$ ...(iii)

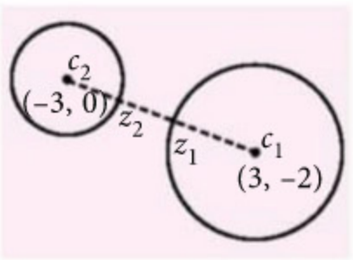
$z^2 - \bar{\alpha} z + \bar{\beta} = 0$ ...(iv)

(iii) – (iv) $\Rightarrow \quad z = \dfrac{\bar{\beta} - \beta}{\alpha + \bar{\alpha}}$

Put in (i) $(\bar{\beta} - \beta)^2 + \alpha(\bar{\beta} - \beta)(\alpha + \bar{\alpha})^2 = 0$

$\Rightarrow \quad (\bar{\beta} - \beta)^2 + (\alpha + \bar{\alpha})(\alpha\bar{\beta} + \alpha\bar{\beta}) = 0$

13. (b, d) : Slope of $c_1c_2 = -\dfrac{1}{3}$



$z_1 = \left(3 + 4\left(\dfrac{-3}{\sqrt{10}}\right), -2 + 4\left(\dfrac{1}{\sqrt{10}}\right)\right)$

$z_2 = \left(-3 - 1\left(-\dfrac{3}{\sqrt{10}}\right), 0 - 1\left(\dfrac{1}{\sqrt{10}}\right)\right)$

14. (b, c, d) : $3|z_4 - z_1| = |z_4 - z_1| + |z_4 - z_2| + |z_4 - z_3|$

Similarly $3|z_4 - z_2| = |z_4 - z_1| + |z_4 - z_2| + |z_4 - z_3|$

$3|z_4 - z_3| = |z_4 - z_1| + |z_4 - z_2| + |z_4 - z_3|$

$\Rightarrow \quad |z_4 - z_1| = |z_4 - z_2| = |z_4 - z_3|$

and $\dfrac{z_4 - z_1}{z_3 - z_2}$ is purely imaginary.

$\Rightarrow$ z_4 represents orthocentre and circumcentre of ΔABC

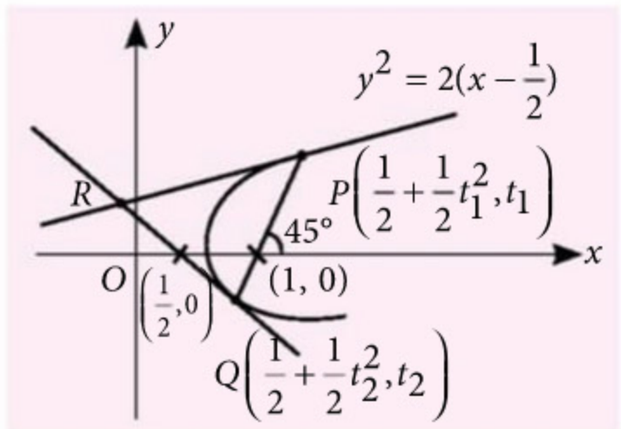
$\Rightarrow$ ΔABC is equilateral.

15. (b, c, d) : $x^2 = (x - 1)^2 + y^2$

$\dfrac{2}{t_1 + t_2} = 1 \quad \Rightarrow \quad t_1 + t_2 = 2$

$R \equiv \left(\dfrac{1}{2}t_1t_2 + \dfrac{1}{2}, \dfrac{1}{2}(t_1 + t_2)\right) = \left(\dfrac{1}{2} + \dfrac{1}{2}t_1t_2, 1\right)$

$\Rightarrow$ R lies on $\text{Im}(z) = 1$

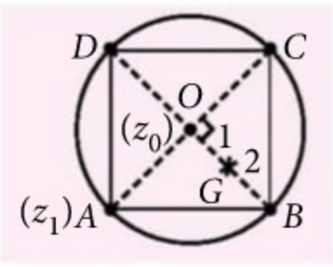


16. (b, c) : $\dfrac{z_G - z_0}{OG} = \dfrac{z_1 - z_0}{OA} e^{\pm i\frac{\pi}{2}}$

Also, $OA = 3OG$

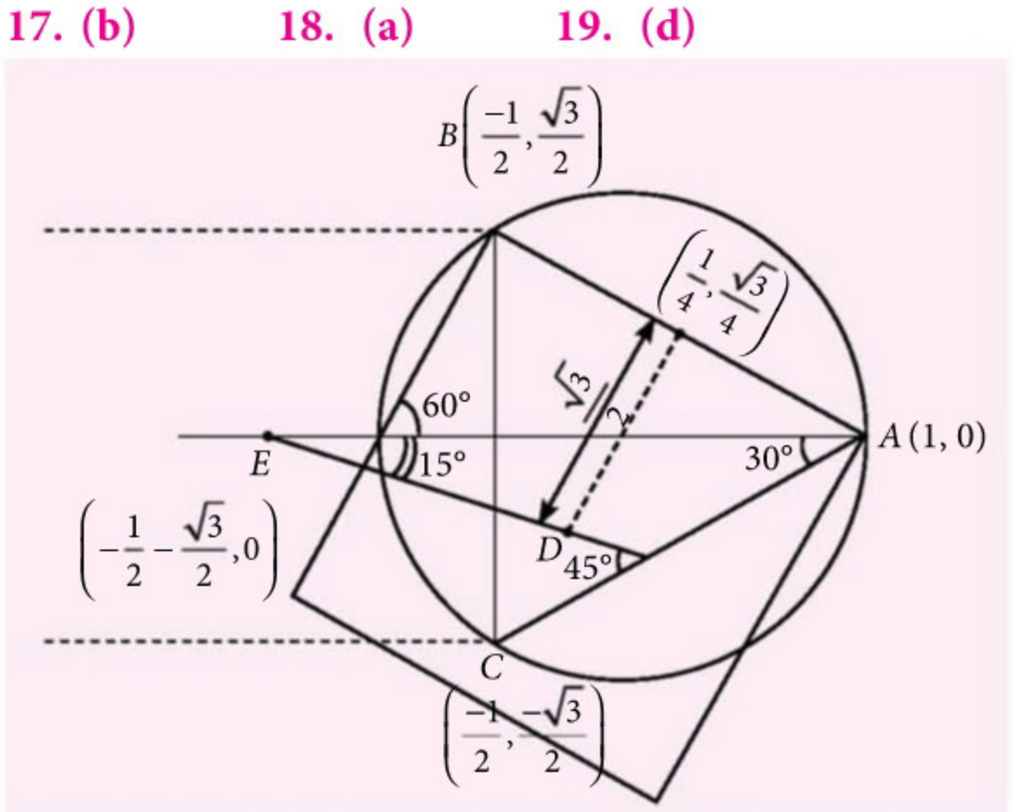
$\Rightarrow \quad z_G - z_0 = \pm i\left(\dfrac{z_1 - z_0}{3}\right)$

$\Rightarrow \quad z_G = z_0\left(1 - \dfrac{i}{3}\right) + \dfrac{i}{3}z_1$ or $z_0\left(1 + \dfrac{i}{3}\right) - \dfrac{i}{3}z_1$



(17-19) :

17. (b) **18. (a)** **19. (d)**



$D \equiv \left(\dfrac{1}{4} - \dfrac{\sqrt{3}}{2}\dfrac{1}{2}, \dfrac{\sqrt{3}}{4} - \dfrac{\sqrt{3}}{2}\dfrac{\sqrt{3}}{2}\right) = \left(\dfrac{1 - \sqrt{3}}{4}, \dfrac{\sqrt{3} - 3}{4}\right)$

Similarly $E \equiv \left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}, 0\right)$

Slope of $DE = \dfrac{\frac{\sqrt{3} - 3}{4}}{\frac{3}{4} + \frac{\sqrt{3}}{4}} = \sqrt{3} - 2$

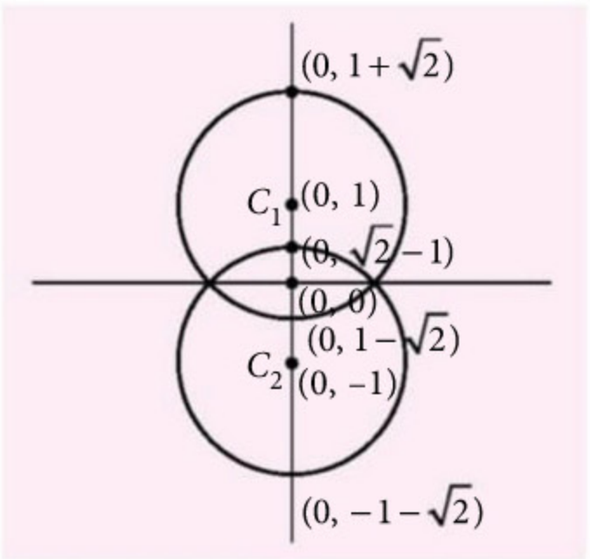
$\Rightarrow$ Angle between AC and $DE = \dfrac{\pi}{4}$

$$DE^2 = \left(\frac{\sqrt{3}-3}{4}\right)^2 + \left(\frac{\sqrt{3}+3}{4}\right)^2 = \frac{2}{16}(3+9) = \frac{3}{2}$$

$$AE = 1 + \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{3+\sqrt{3}}{2}$$

(20-22) :

20. (d) **21. (a)** **22. (b)**



$$|z^2 + 1|^2 = 4|z|^2$$

$$\Rightarrow (z^2 + 1)(\bar{z}^2 + 1) = 4|z|^2$$

$$\Rightarrow (|z|^4 - 2|z|^2 + 1) + (z^2 + \bar{z}^2 - 2|z|^2) = 0$$

$$\Rightarrow (|z|^2 - 1)^2 - (i(z - \bar{z}))^2 = 0$$

$$\Rightarrow (z\bar{z} - i(z - \bar{z}) - 1)(z\bar{z} + i(z - \bar{z}) - 1) = 0$$

$$|z|_{max} = 1 + \sqrt{2}$$

If $z = re^{i\frac{\pi}{6}}$, then

From $z\bar{z} - i(z - \bar{z}) - 1 = 0$

$$\Rightarrow r^2 + r - 1 = 0 \Rightarrow r = \frac{-1+\sqrt{5}}{2}$$

and from $z\bar{z} + i(z - \bar{z}) - 1 = 0$

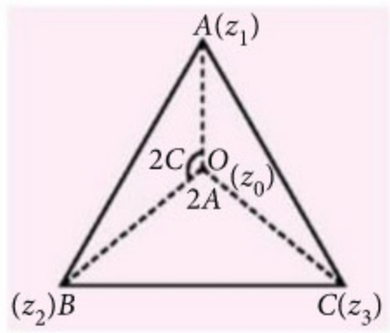
$$\Rightarrow r^2 - r - 1 = 0 \Rightarrow r = \frac{1+\sqrt{5}}{2}$$

Also $C_1C_2^2 = 4 = r_1^2 + r_2^2 = (\sqrt{2})^2 + (\sqrt{2})^2$

$\Rightarrow$ Circles are orthogonal

23. (1) : $(z_0 - z_2) = (z_0 - z_1)e^{i2C}$

$(z_0 - z_3) = (z_0 - z_2)e^{i2A}$



$$\therefore \left(\frac{z_0 - z_1}{z_0 - z_2}\right)\frac{\sin 2A}{\sin 2B} + \left(\frac{z_0 - z_3}{z_0 - z_2}\right)\frac{\sin 2C}{\sin 2B}$$

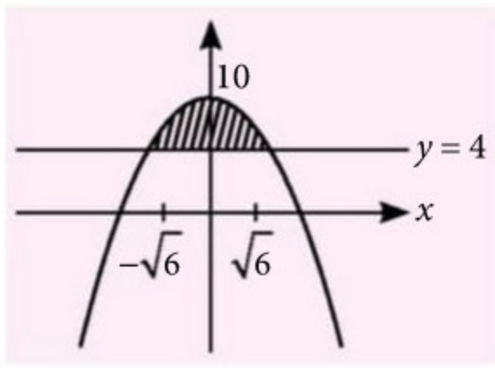
$$= \frac{(\cos 2C - i\sin 2C)\sin 2A + (\cos 2A + i\sin 2A)\sin 2C}{\sin 2B}$$

$$= \frac{\sin 2(A+C)}{\sin 2B} = -1$$

24. (384) : $y \geq 4$

and $x^2 + 4 + y - 4 \leq 10$

$\Rightarrow y \leq 10 - x^2$



Area

$$= 2\left(\int_0^{\sqrt{6}} (10 - x^2)dx - 4\sqrt{6}\right)$$

$$= 2\left(10\sqrt{6} - \frac{6\sqrt{6}}{3} - 4\sqrt{6}\right) = 8\sqrt{6}$$

25. (18) : Let $z_1 = e^{i\theta_1}$, $z_2 = 2e^{i\theta_2}$, $z_3 = 3e^{i\theta_3}$

$\Rightarrow z_1 + z_2 + z_3 = 0$

$\Rightarrow z_1^3 + z_2^3 + z_3^3 = 3z_1z_2z_3$

$\Rightarrow e^{i3\theta_1} + 8e^{i3\theta_2} + 27e^{i3\theta_3} = 18e^{i(\theta_1 + \theta_2 + \theta_3)}$

$\Rightarrow \sin 3\theta_1 + 8\sin 3\theta_2 + 27\sin 3\theta_3 = 18\sin(\theta_1 + \theta_2 + \theta_3)$

$\Rightarrow \lambda = 18$

26. (8) : $z^2 - \sqrt{3}z + 1 = 0 \Rightarrow z = \frac{\sqrt{3} \pm i}{2} \Rightarrow z = e^{i\frac{\pi}{6}}$ or $e^{-i\frac{\pi}{6}}$

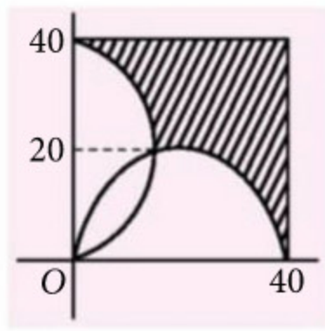
$$\Rightarrow z^r + \frac{1}{z^r} = 2\cos\frac{r\pi}{6} \Rightarrow \sum_{r=1}^{5}\left(z^r + \frac{1}{z^r}\right)^2 = 4\sum_{r=1}^{4}\cos^2\left(\frac{r\pi}{6}\right)$$

$$= 2\sum_{r=1}^{5}\left(1 + \cos\frac{2r\pi}{6}\right) = 2(5 + (-1)) = 8$$

27. (571) : $0 \leq x \leq 40$, $0 \leq y \leq 40$

and $\frac{40}{\bar{z}} = \frac{40(x+iy)}{x^2+y^2}$

$$\Rightarrow 0 \leq \frac{40x}{x^2+y^2} \leq 1$$



$\Rightarrow 40x \leq x^2 + y^2$

and $0 \leq \frac{40y}{x^2+y^2} \leq 1$

$\Rightarrow 40y \leq x^2 + y^2$

$A = (40)^2 - [\pi(20)^2 - \{(20)^2 - 2((20)^2 - \frac{\pi}{4}(20)^2)\}]$

$$\Rightarrow A = (20)^2\left(3 - \frac{\pi}{2}\right) \approx 571.68$$

$\Rightarrow [A] = 571$

Monthly Test Drive-3 CLASS XII ANSWER KEY

1. (d)	**2.** (b)	**3.** (a)	**4.** (d)	**5.** (d)
6. (b)	**7.** (a, c)	**8.** (b, c, d)	**9.** (d)	**10.** (a, c, d)
11. (a, b, c)	**12.** (c)	**13.** (a, c, d)	**14.** (d)	**15.** (d)
16. (a)	**17.** (1)	**18.** (1)	**19.** (2)	**20.** (4)

CBSE warm-up!

CLASS-XII

Chapterwise Practice questions for CBSE Exams as per the reduced syllabus and marking scheme issued by CBSE for the academic session 2020-21.

Series 3 **Matrices and Determinants**

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) This question paper contains **36** questions.
(iii) Question **1-20** in Section-A are very short-answer-objective type questions carrying **1** mark each.
(iv) Question **21-26** in Section-B are short-answer type questions carrying **2** marks each.
(v) Question **27-32** in Section-C are long-answer-**I** type questions carrying **4** marks each.
(vi) Question **33-36** in Section-D are long-answer-**II** type questions carrying **6** marks each.

SECTION-A

(Q.1 - Q.10) are multiple choice type questions. Select the correct option.

1. If the elements a_{ij} of a 3×3 matrix are given by $a_{ij} = \frac{1}{2}|-3i + j|$. Then, the value of element a_{32} is

(a) $\frac{3}{2}$ (b) $\frac{7}{2}$ (c) $\frac{5}{2}$ (d) $\frac{9}{2}$

2. If $\begin{bmatrix} 2x+1 & 2y \\ 0 & y^2+1 \end{bmatrix} = \begin{bmatrix} x+3 & 10 \\ 0 & 26 \end{bmatrix}$, then find the value of $(x + y)$.

(a) 8 (b) 5 (c) 7 (d) 9

3. If $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$, find the value of $3|A|$.

(a) 9 (b) 6 (c) 12 (d) 15

4. If A is a skew-symmetric matrix of order 3, then the value of $|A|$ is

(a) 3 (b) 0 (c) 9 (d) 27

5. If $A = \begin{bmatrix} 5 & 6 & -3 \\ -4 & 3 & 2 \\ -4 & -7 & 3 \end{bmatrix}$, then write the cofactor of the element a_{21}.

(a) 3 (b) 4 (c) 2 (d) 5

6. If $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 3 \\ 3 & 3 & 5 \end{bmatrix}$, then find A (adj A).

(a) $4I$ (b) $2I$ (c) $5I$ (d) $7I$

7. For what value of x, is the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$, a skew-symmetric matrix?

(a) 2 (b) -1 (c) 4 (d) -3

8. If $(2x \quad 4)\begin{pmatrix} x \\ -8 \end{pmatrix} = O$, find the positive value of x.

(a) 2 (b) 3 (c) 5 (d) 4

9. If $2\begin{bmatrix}1 & 3\\ 0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$, then write the value of $(x+y)$.

(a) 5 (b) 6 (c) 4 (d) 8

10. Find (adj A), if $A=\begin{bmatrix}5 & 2\\ 7 & 3\end{bmatrix}$.

(a) $\begin{bmatrix}3 & -2\\ -7 & 5\end{bmatrix}$ (b) $\begin{bmatrix}3 & 2\\ 7 & 5\end{bmatrix}$

(c) $\begin{bmatrix}3 & -7\\ -2 & 5\end{bmatrix}$ (d) none of these

(Q. 11-Q.15) Fill in the blanks.

11. If $[2\ 1\ 3]\begin{bmatrix}-1 & 0 & -1\\ -1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}=A$, then the order of matrix A is _____.

12. If $\begin{bmatrix}2 & 3\\ 5 & 7\end{bmatrix}\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}=\begin{bmatrix}-4 & 6\\ -9 & x\end{bmatrix}$, then the value of x is _____.

OR

_____ matrix is both symmetric and skew symmetric matrix.

13. If A is a matrix of order 3×3, then $(A^2)^{-1}=$ _____.

14. If A and B are symmetric matrices of same order, then AB is symmetric if and only if _____.

OR

If a matrix has 5 elements, then number of possible orders that such matrix can have, is _____.

15. If a is a matrix of order 3×2, then the order of the matrix A' is _____.

(Q.16-Q.20) Answer the following questions.

16. Find the value of k in order that the points (5, 5), (k, 1) and (10, 7) are collinear.

17. Construct a 3×2 matrix whose elements are given by $a_{ij}=(i+2j)$.

18. Prove that $\begin{vmatrix}1 & a & b+c\\ 1 & b & c+a\\ 1 & c & a+b\end{vmatrix}=0$.

19. Find the value of x and y, given that $\begin{bmatrix}x & y\\ 3y & x\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}=\begin{bmatrix}3\\ 5\end{bmatrix}$.

OR

Find x, y, z, w if $\begin{bmatrix}x-y & 2x+z\\ 2x-y & 3z+w\end{bmatrix}=\begin{bmatrix}-1 & 5\\ 0 & 13\end{bmatrix}$.

20. If $A^T=\begin{bmatrix}3 & 4\\ -1 & 2\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}$, then find A^T-B^T.

SECTION-B

21. If $2\begin{bmatrix}3 & 4\\ 5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\ 10 & 5\end{bmatrix}$, find $(x-y)$.

22. Given $A=\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$, compute A^{-1}.

OR

If $A=\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}$, find the determinant of the matrix A^2-2A.

23. Determine the values of x for which the matrix $A=\begin{bmatrix}x+1 & -3 & 4\\ -5 & x+2 & 2\\ 4 & 1 & x-6\end{bmatrix}$ is singular.

24. If $A=\begin{bmatrix}2 & -1\\ 3 & 4\\ 1 & 5\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 3\\ 2 & 1\end{bmatrix}$, find AB.

Does BA exist ?

OR

In the interval $\pi/2<x<\pi$, find the value of x for which the matrix $\begin{bmatrix}2\sin x & 3\\ 1 & 2\sin x\end{bmatrix}$ is singular.

25. If $A=\begin{bmatrix}3 & 5\\ 7 & -9\end{bmatrix}$ and $B=\begin{bmatrix}6 & -4\\ 2 & 3\end{bmatrix}$, then find $4A-3B$.

26. Find a matrix A such that $2A-3B+5C=O$, where $B=\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$.

SECTION-C

27. Let $A=\begin{bmatrix}1 & -2\\ 5 & 4\\ 3 & 0\end{bmatrix}$, $B=\begin{bmatrix}3 & 1\\ 0 & 2\\ -3 & 5\end{bmatrix}$ and $C=\begin{bmatrix}4 & 3\\ -2 & 2\\ 1 & 6\end{bmatrix}$.

Verify that $(A+B)+C=A+(B+C)$.

28. If $A=\begin{bmatrix}2 & 3\\ 1 & -4\end{bmatrix}, B=\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

OR

If $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$ and I is the identity matrix of order 2, then show that $A^2 = 4A - 3I$. Hence find A^{-1}.

29. Find the equation of the line joining $A(1, 3)$ and $B(0, 0)$ using determinants and find the value of k if $D(k, 0)$ is a point such that area of ΔABD is 3 square units.

30. If the coordinates of the vertices of an equilateral triangle with sides of length 'a' are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then prove that

$$\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}^2 = \frac{3a^4}{4}.$$

31. Find the adjoint of the matrix $A = \begin{bmatrix}-1 & -2 & -2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$ and hence show that $A\cdot(\text{adj } A) = |A|I_3$.

OR

If $A=\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 4\\ -2 & 2 & 1\end{bmatrix}$, then find $(A')^{-1}$.

32. If $A=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, then find the values of a and b.

SECTION-D

33. If $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, find x and y such that $(xI + yA)^2 = A$.

34. If $A=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$, find adj A and verify that $A(\text{adj } A) = (\text{adj } A)A = |A|\, I_3$.

OR

Let $A=\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}$ and $f(x)= x^2 - 4x + 7$. Show that $f(A) = O$. Use this result to find A^5.

35. Find the matrix A such that $|A| = 2$ and $\text{adj } A=\begin{bmatrix}2 & 2 & 0\\ 2 & 5 & 1\\ 0 & 1 & 1\end{bmatrix}$.

OR

Find A^{-1}, where $A=\begin{bmatrix}1 & 2 & -3\\ 2 & 3 & 2\\ 3 & -3 & -4\end{bmatrix}$. Hence solve the system of equations
$x + 2y - 3z = -4$, $2x + 3y + 2z = 2$, $3x - 3y - 4z = 11$.

36. Let $A=\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}$ and I is the identity matrix of order 2.

Show that $(I + A) = (I - A)\cdot\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$.

SOLUTIONS

1. **(b) :** Here, $a_{ij} = \frac{1}{2}|-3i + j|$

$\therefore\quad a_{32} = \frac{1}{2}|-3\cdot 3 + 2|$ [For $i = 3, j = 2$]

$= \frac{1}{2}|-9 + 2| = \frac{1}{2}|-7| = \frac{7}{2}$

2. **(c) :** Given, $\begin{bmatrix}2x+1 & 2y\\ 0 & y^2+1\end{bmatrix} = \begin{bmatrix}x+3 & 10\\ 0 & 26\end{bmatrix}$

$\Rightarrow\quad 2x + 1 = x + 3,\ 2y = 10,\ y^2 + 1 = 26$

From first two equations, we get $x = 2, y = 5$ which also satisfies $y^2 + 1 = 26$

$\therefore\quad x + y = 2 + 5 = 7$

3. **(b) :** Here, $A=\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix}$

$\Rightarrow\quad |A| = \begin{vmatrix}3 & 4\\ 1 & 2\end{vmatrix} = 3\times 2 - 1\times 4 = 2 \Rightarrow 3|A| = 3\times 2 = 6.$

4. **(b) :** Let $A=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}$ be any skew-symmetric matrix of order 3, where a, b, c are arbitrary real numbers.

Then, $|A| = \begin{vmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{vmatrix}$

$= 0\cdot(0 + c^2) - a\cdot(0 + bc) + b\cdot(ac - 0)$

$= 0 - abc + abc = 0.$

5. **(a) :** We have, $A=\begin{bmatrix}5 & 6 & -3\\ -4 & 3 & 2\\ -4 & -7 & 3\end{bmatrix}$

$\therefore$ Cofactor of $a_{21} = (-1)^{2+1}\begin{vmatrix}6 & -3\\ -7 & 3\end{vmatrix} = -1\,(18-21) = 3$

6. **(b) :** Given, $A = \begin{bmatrix}2 & 0 & 0\\ -1 & 2 & 3\\ 3 & 3 & 5\end{bmatrix}$

We know, $A(\text{adj } A) = |A|\, I = \begin{vmatrix}2 & 0 & 0\\ -1 & 2 & 3\\ 3 & 3 & 5\end{vmatrix} I$

$= (2(10-9) - 0 + 0)\, I = 2I$

7. **(a) :** The matrix $A=\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & 3\\ x & -3 & 0\end{bmatrix}$ is a skew-symmetric matrix.

$\therefore \quad A' = -A \Rightarrow \begin{bmatrix}0 & -1 & x\\ 1 & 0 & -3\\ -2 & 3 & 0\end{bmatrix} = \begin{bmatrix}0 & -1 & 2\\ 1 & 0 & -3\\ -x & 3 & 0\end{bmatrix} \Rightarrow x = 2$

8. **(d) :** Here, $(2x \quad 4)\begin{pmatrix}x\\ -8\end{pmatrix} = O$

$\Rightarrow \quad 2x \cdot x + 4 \cdot (-8) = 0 \Rightarrow 2x^2 - 32 = 0$

$\Rightarrow \quad x^2 = 16 = 4^2 \Rightarrow x = 4$

which is the required positive value of x.

9. **(b) :** Given, $2\begin{bmatrix}1 & 3\\ 0 & x\end{bmatrix} + \begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix} = \begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix}2 & 6\\ 0 & 2x\end{bmatrix} + \begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix} = \begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix}2+y & 6\\ 1 & 2x+2\end{bmatrix} = \begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$

$\Rightarrow \quad 2 + y = 5,\ 2x + 2 = 8 \Rightarrow y = 3, x = 3$

$\therefore \quad x + y = 6$

10. **(a) :** Here, $A=\begin{bmatrix}5 & 2\\ 7 & 3\end{bmatrix}$

Cofactor matrix of $A=\begin{bmatrix}3 & -7\\ -2 & 5\end{bmatrix}$

$\therefore \quad \text{adj } A = \begin{bmatrix}3 & -7\\ -2 & 5\end{bmatrix}' = \begin{bmatrix}3 & -2\\ -7 & 5\end{bmatrix}$

11. Given, $A = [2 \quad 1 \quad 3]\begin{bmatrix}-1 & 0 & -1\\ -1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$

$= [-2-1 \quad 1+3 \quad -2+3]\begin{bmatrix}1\\ 0\\ -1\end{bmatrix} = [-3 \quad 4 \quad 1]\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$

$= [-3 + 0 - 1] = [-4]$

$\therefore \quad$ The order of matrix $A = 1 \times 1$

12. Given, $\begin{bmatrix}2 & 3\\ 5 & 7\end{bmatrix}\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix} = \begin{bmatrix}-4 & 6\\ -9 & x\end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix}2-6 & -6+12\\ 5-14 & -15+28\end{bmatrix} = \begin{bmatrix}-4 & 6\\ -9 & x\end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix}-4 & 6\\ -9 & 13\end{bmatrix} = \begin{bmatrix}-4 & 6\\ -9 & x\end{bmatrix} \Rightarrow x = 13$

OR

Null matrix is both symmetric and skew-symmetric matrix.

13. We have, $(A^2)^{-1} = (A \cdot A)^{-1}$

$= A^{-1} \cdot A^{-1} \quad [\because (AB)^{-1} = B^{-1}A^{-1}]$

$= (A^{-1})^2$

14. Given A and B are symmetric matrices

$\therefore \quad A' = A$ and $B' = B \Rightarrow (AB)' = B'A' = BA$

Now, AB is symmetric if $(AB)' = AB$ for which we must have $AB = BA$.

OR

The possible orders are 1×5 or 5×1.

So, given matrix can have 2 orders.

15. If A is a matrix of order 3×2, then the order of the matrix A' is 2×3.

16. The given points are collinear.

$\therefore \quad \begin{vmatrix}5 & 5 & 1\\ k & 1 & 1\\ 10 & 7 & 1\end{vmatrix} = 0$

$\Rightarrow \quad 5(1-7) - 5(k-10) + 1(7k-10) = 0$

$\Rightarrow \quad -30 - 5k + 50 + 7k - 10 = 0 \Rightarrow 2k + 10 \Rightarrow k = -5$

17. $a_{ij} = (i + 2j)$ for $i = 1, 2, 3$ and $j = 1, 2$

$\therefore \quad a_{11} = (1 + 2 \times 1) = 3\ ;\ a_{12} = (1 + 2 \times 2) = 5\ ;$

$a_{21} = (2 + 2 \times 1) = 4\ ;\ a_{22} = (2 + 2 \times 2) = 6\ ;$

$a_{31} = (3 + 2 \times 1) = 5\ ;\ a_{32} = (3 + 2 \times 2) = 7.$

Hence, the required matrix $A=\begin{bmatrix}3 & 5\\ 4 & 6\\ 5 & 7\end{bmatrix}$

18. We have, L.H.S. $= \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

$= 1(ab + b^2 - c^2 - ac) - 1(a^2 + ab - bc - c^2)$
$\quad + 1(ac + a^2 - b^2 - bc)$
$= ab + b^2 - c^2 - ac - a^2 - ab + bc + c^2 + ac + a^2 - b^2 - bc$
$= 0 =$ R.H.S.

19. We have, $\begin{bmatrix} x & y \\ 3y & x \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x+2y \\ 3y+2x \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\Rightarrow \quad x + 2y = 3 \quad ...(i) \qquad 2x + 3y = 5 \quad ...(ii)$

Solving (i) and (ii), we get $y = 1, x = 1$

OR

Given $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

$\Rightarrow \quad x - y = -1, 2x - y = 0, 2x + z = 5$ and $3z + w = 13$

Solving the first two equations, we get $x = 1$ and $y = 2$

Putting $x = 1$ in $2x + z = 5$, we get $z = 3$

Putting $z = 3$ in $3z + w = 13$, we get $w = 4$

$\therefore \quad x = 1, \ y = 2, z = 3$ and $w = 4$

20. Given, $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$\Rightarrow \quad B^T = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

$\therefore \quad A^T - B^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$

21. We have, $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 6 & 8 \\ 10 & 2x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

$\Rightarrow \quad 2x + 1 = 5$ and $8 + y = 0 \Rightarrow x = 2$ and $y = -8$

$\therefore \quad x - y = 2 - (-8) = 10$

22. We have, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix} = 14 - 12 = 2 \neq 0$

So, A is a non-singular matrix and therefore it is invertible.

$\therefore \quad \text{adj } A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

Hence, $A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

OR

$A^2 - 2A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+6 & 3+3 \\ 2+2 & 6+1 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$\therefore \quad |A^2 - 2A| = \begin{vmatrix} 5 & 0 \\ 0 & 5 \end{vmatrix} = 25 - 0 = 25$

23. For the matrix A to be singular $|A| = 0$

$\Rightarrow \begin{vmatrix} x+1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{vmatrix} = 0$

$\Rightarrow \quad (x + 1)\{(x + 2)(x - 6) - 2\} + 3\{-5x + 30 - 8\} +$
$\qquad 4\{-5 - 4x - 8\} = 0$

$\Rightarrow \quad (x + 1)(x^2 - 4x - 14) + 3(-5x + 22) + 4(-4x - 13) = 0$

$\Rightarrow \quad x(x^2 - 3x - 49) = 0$

$\Rightarrow \quad x = 0, \ x = \frac{1}{2}(3 \pm \sqrt{205})$

24. Here, A is a 3×2 matrix and B is a 2×2 matrix. Clearly, the number of columns in A equals the number of rows in B.

$\therefore \quad AB$ exists and it is a 3×2 matrix.

Now, $AB = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2-2 & 6-1 \\ -3+8 & 9+4 \\ -1+10 & 3+5 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ 5 & 13 \\ 9 & 8 \end{bmatrix}$

Further, B is a 2×2 matrix and A is a 3×2 matrix. So, the number of columns in B is not equal to the number of rows in A. So, BA does not exist.

OR

For the matrix $\begin{bmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{bmatrix}$ to be singular, its determinant $= 0$

$\therefore \quad \begin{vmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{vmatrix} = 0$

$\Rightarrow \quad 4\sin^2 x - 3 = 0 \Rightarrow \sin^2 x = \frac{3}{4} \Rightarrow \sin x = \pm\frac{\sqrt{3}}{2}$

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$$\therefore \quad x=\frac{2\pi}{3}\left(\because \frac{\pi}{2}<x<\pi\right)$$

25. We have, $4A-3B=4A+(-3B)$

$$=\begin{bmatrix}12 & 20\\ 28 & -36\end{bmatrix}+\begin{bmatrix}-18 & 12\\ -6 & -9\end{bmatrix}$$

$$=\begin{bmatrix}12+(-18) & 20+12\\ 28+(-6) & -36+(-9)\end{bmatrix}=\begin{bmatrix}-6 & 32\\ 22 & -45\end{bmatrix}$$

26. Let $A=\begin{bmatrix}x & y & z\\ p & q & r\end{bmatrix}$ [$\because B$ and C are 2×3 matrices]

Given, $2A-3B+5C=O$

$$\Rightarrow \quad 2A=3B-5C \quad \Rightarrow \quad A=\frac{1}{2}[3B-5C] \qquad ...(i)$$

Now, $3B-5C=3\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}-5\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$

$$=\begin{bmatrix}-6 & 6 & 0\\ 9 & 3 & 12\end{bmatrix}-\begin{bmatrix}10 & 0 & -10\\ 35 & 5 & 30\end{bmatrix}=\begin{bmatrix}-16 & 6 & 10\\ -26 & -2 & -18\end{bmatrix}$$

From (i), we get $A=\begin{bmatrix}-8 & 3 & 5\\ -13 & -1 & -9\end{bmatrix}$.

27. Clearly, each one of the matrices A, B, C is a (3×2) matrix. So, $(A+B)+C$ and $A+(B+C)$ are both defined and each one is a 3×2 matrix.

Now, $(A+B)=\begin{bmatrix}1 & -2\\ 5 & 4\\ 3 & 0\end{bmatrix}+\begin{bmatrix}3 & 1\\ 0 & 2\\ -3 & 5\end{bmatrix}=\begin{bmatrix}4 & -1\\ 5 & 6\\ 0 & 5\end{bmatrix}$

$$\therefore \quad (A+B)+C=\begin{bmatrix}4 & -1\\ 5 & 6\\ 0 & 5\end{bmatrix}+\begin{bmatrix}4 & 3\\ -2 & 2\\ 1 & 6\end{bmatrix}=\begin{bmatrix}8 & 2\\ 3 & 8\\ 1 & 11\end{bmatrix}$$

Also, $(B+C)=\begin{bmatrix}3 & 1\\ 0 & 2\\ -3 & 5\end{bmatrix}+\begin{bmatrix}4 & 3\\ -2 & 2\\ 1 & 6\end{bmatrix}=\begin{bmatrix}7 & 4\\ -2 & 4\\ -2 & 11\end{bmatrix}$

$$\therefore \quad A+(B+C)=\begin{bmatrix}1 & -2\\ 5 & 4\\ 3 & 0\end{bmatrix}+\begin{bmatrix}7 & 4\\ -2 & 4\\ -2 & 11\end{bmatrix}=\begin{bmatrix}8 & 2\\ 3 & 8\\ 1 & 11\end{bmatrix}$$

Hence, $(A+B)+C=A+(B+C)$.

28. Refer to answer 97, page no. 70 (MTG CBSE Champion Mathematics Class 12).

OR

$$A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$$

$$\Rightarrow \quad A^2=\begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix} \qquad ...(i)$$

Now, $4A-3I=4\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}-3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$$\Rightarrow \quad 4A-3I=\begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix} \qquad ...(ii)$$

From (i) and (ii), we get $A^2=4A-3I$

Pre-multiplying by A^{-1} on both sides, we get

$A^{-1}(A^2)=4A^{-1}A-3A^{-1}I$

$\Rightarrow \quad A=4I-3A^{-1}$ [$\because A^{-1}A=I$]

$\Rightarrow \quad 3A^{-1}=4I-A$

$$\Rightarrow \quad A^{-1}=\frac{4}{3}I-\frac{1}{3}A$$

$$=\frac{4}{3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\frac{1}{3}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}2/3 & 1/3\\ 1/3 & 2/3\end{bmatrix}$$

29. Refer to answer 66, page no. 67 (MTG CBSE Champion Mathematics Class 12).

30. As we know area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3), is given by

$$\Delta=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$$

$$\Rightarrow \quad \Delta^2=\frac{1}{4}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}^2 \qquad ...(i)$$

Since, area of an equilateral triangle with side of length a is $\Delta=\frac{\sqrt{3}}{4}a^2$

$$\Rightarrow \quad \Delta^2=\frac{3}{16}a^4 \qquad ...(ii)$$

From (i) and (ii), we get

$$\frac{3}{16}a^4=\frac{1}{4}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}^2$$

$$\Rightarrow \quad \begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}^2=\frac{3}{4}a^4.$$

31. Here, $A=\begin{bmatrix}-1 & -2 & -2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix} \Rightarrow |A|=\begin{vmatrix}-1 & -2 & -2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{vmatrix}$

$=-1(1-4)-(-2)(2+4)-2(-4-2)=3+12+12=27$

Also, $A_{11} = -3, A_{12} = -6, A_{13} = -6,$
$A_{21} = 6, A_{22} = 3, A_{23} = -6,$
$A_{31} = 6, A_{32} = -6, A_{33} = 3$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$\text{Now, } A \cdot (\text{adj } A) = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} = 27 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3.$$

OR

Refer to answer 93, page no. 69 (MTG CBSE Champion Mathematics Class 12).

32. We have, $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$

$$\text{Consider, } (A + B) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$$

$$\therefore \quad (A + B)^2 = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix} \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(1+a-2) & 4 \end{bmatrix} = \begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(a-1) & 4 \end{bmatrix}$$

$$\text{Now, consider } A^2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & -1+1 \\ 2-2 & -2+1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\text{and } B^2 = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix}$$

$$\therefore \quad A^2 + B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$$

It is given that $(A + B)^2 = A^2 + B^2$

$$\therefore \quad \begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(a-1) & 4 \end{bmatrix} = \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$$

By equality of matrices, comparing the corresponding elements, we get
$a - 1 = 0 \Rightarrow a = 1$ and $b = 4$

Also, $(1 + a)^2 = a^2 + b - 1$ and $(2 + b)(a - 1) = ab - b$
satisfied by $a = 1$ and $b = 4$
Therefore, $a = 1$ and $b = 4$.

33. We have, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\therefore \quad xI + yA = x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad xI + yA = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

Now, $(xI + yA)^2 = A$ (Given)

$$\Rightarrow \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \begin{bmatrix} x & y \\ -y & x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x^2 - y^2 & 2xy \\ -2xy & x^2 - y^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\Rightarrow x^2 - y^2 = 0, 2xy = 1, -2xy = -1$ and $x^2 - y^2 = 0$
$\Rightarrow x^2 - y^2 = 0$ and $2xy = 1$. Now two cases arise.

Case I : When $x = y$ and $2xy = 1$

$$\Rightarrow \quad 2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

$$\therefore \quad x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}} \text{ or } x = -\frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}$$

Case II : When $x = -y$ and $2xy = 1$

$$\Rightarrow \quad -2x^2 = 1 \Rightarrow x = \pm \frac{i}{\sqrt{2}}$$

$$\therefore \quad x = \frac{i}{\sqrt{2}}, y = -\frac{i}{\sqrt{2}} \text{ or } x = -\frac{i}{\sqrt{2}}, y = \frac{i}{\sqrt{2}}$$

34. Refer to answer 98, page no. 70 (MTG CBSE Champion Mathematics Class 12).

OR

We have, $f(x) = x^2 - 4x + 7$

$$\therefore \quad f(A) = A^2 - 4A + 7I_2$$

$$\text{Here, } A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$\therefore \quad A^2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$$

$$-4A = \begin{bmatrix} -8 & -12 \\ 4 & -8 \end{bmatrix} \text{ and } 7I_2 = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

Now, $f(A) = A^2 - 4A + 7I_2$

$$\Rightarrow \quad f(A) = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} -8 & -12 \\ 4 & -8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\Rightarrow \quad f(A)=\begin{bmatrix}1-8+7 & 12-12+0\\ -4+4+0 & 1-8+7\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$$

Now, $f(A) = O$

$\Rightarrow \quad A^2 - 4A + 7I_2 = O \Rightarrow A^2 = 4A - 7I_2$

$\Rightarrow \quad A^3 = 4A^2 - 7I_2A \Rightarrow A^3 = 4(4A - 7I_2) - 7A$

$\Rightarrow \quad A^3 = 9A - 28I_2$

$\Rightarrow \quad A^4 = 9A^2 - 28I_2A = 9(4A - 7I_2) - 28A$

$\Rightarrow \quad A^4 = 36A - 63I_2 - 28A = 8A - 63I_2$

$\Rightarrow \quad A^5 = 8A^2 - 63I_2A$

$\Rightarrow \quad A^5 = 8(4A - 7I_2) - 63A = -31A - 56I_2$

$$\Rightarrow \quad A^5 = -31\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix} - 56\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}-62 & -93\\ 31 & -62\end{bmatrix} + \begin{bmatrix}-56 & 0\\ 0 & -56\end{bmatrix} = \begin{bmatrix}-118 & -93\\ 31 & -118\end{bmatrix}.$$

35. We know that $A(\text{adj } A) = |A| I$

$\Rightarrow \quad A = |A| I (\text{adj } A)^{-1}$

$\Rightarrow \quad A = 2 (\text{adj } A)^{-1} \qquad [\because |A| = 2]$

$\Rightarrow \quad A = 2B^{-1}$, where $B = \text{adj } A$

$$\text{Now, } B = \text{adj } A = \begin{bmatrix}2 & 2 & 0\\ 2 & 5 & 1\\ 0 & 1 & 1\end{bmatrix}$$

$\therefore \quad |B| = 2(5 - 1) - 2(2 - 0) + 0 \cdot (2 - 0)$

$= 8 - 4 + 0 = 4$

Let C_{ij} be the cofactor of b_{ij} in matrix B. Then,
$C_{11} = 4, C_{12} = -2, C_{13} = 2, C_{21} = -2, C_{22} = 2,$
$C_{23} = -2, C_{31} = 2, C_{32} = -2$ and $C_{33} = 6$

$$\therefore \quad \text{adj } B = \begin{bmatrix}4 & -2 & 2\\ -2 & 2 & -2\\ 2 & -2 & 6\end{bmatrix}^T = \begin{bmatrix}4 & -2 & 2\\ -2 & 2 & -2\\ 2 & -2 & 6\end{bmatrix}$$

$$\text{and } B^{-1} = \frac{1}{|B|}(\text{adj } B) = \frac{1}{4}\begin{bmatrix}4 & -2 & 2\\ -2 & 2 & -2\\ 2 & -2 & 6\end{bmatrix}$$

$$\text{Hence, } A = 2(B)^{-1} = \frac{1}{2}\begin{bmatrix}4 & -2 & 2\\ -2 & 2 & -2\\ 2 & -2 & 6\end{bmatrix} = \begin{bmatrix}2 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 3\end{bmatrix}$$

OR

$$|A| = \begin{vmatrix}1 & 2 & -3\\ 2 & 3 & 2\\ 3 & -3 & -4\end{vmatrix} = -6 + 28 + 45 = 67 \neq 0$$

So, A is invertible.

$$\text{Clearly, adj } A = \begin{bmatrix}-6 & 14 & -15\\ 17 & 5 & 9\\ 13 & -8 & -1\end{bmatrix}^T = \begin{bmatrix}-6 & 17 & 13\\ 14 & 5 & -8\\ -15 & 9 & -1\end{bmatrix}$$

$$\text{So, } \quad A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{67}\begin{bmatrix}-6 & 17 & 13\\ 14 & 5 & -8\\ -15 & 9 & -1\end{bmatrix}$$

The given system of equations is
$x + 2y - 3z = -4, 2x + 3y + 2z = 2, 3x - 3y - 4z = 11.$
The given system of equation can be written as $AX = B$,

$$\text{where } A = \begin{bmatrix}1 & 2 & -3\\ 2 & 3 & 2\\ 3 & -3 & -4\end{bmatrix}, \ X = \begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B = \begin{bmatrix}-4\\ 2\\ 11\end{bmatrix}$$

Since A^{-1} exists, so system of equations has a unique solution given by $X = A^{-1} B$

$$\Rightarrow \quad \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \frac{1}{67}\begin{bmatrix}-6 & 17 & 13\\ 14 & 5 & -8\\ -15 & 9 & -1\end{bmatrix}\begin{bmatrix}-4\\ 2\\ 11\end{bmatrix} = \frac{1}{67}\begin{bmatrix}201\\ -134\\ 67\end{bmatrix} = \begin{bmatrix}3\\ -2\\ 1\end{bmatrix}$$

$\Rightarrow \quad x = 3, y = -2$ and $z = 1$ is the required solution.

36. Let $\tan\frac{\alpha}{2} = t$.

$$\text{Then, } \cos\alpha = \frac{1 - \tan^2(\alpha/2)}{1 + \tan^2(\alpha/2)} = \frac{1 - t^2}{1 + t^2}$$

$$\text{and } \sin\alpha = \frac{2\tan(\alpha/2)}{1 + \tan^2(\alpha/2)} = \frac{2t}{1 + t^2}$$

$$\text{Also, } (I + A) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}0 & -t\\ t & 0\end{bmatrix} = \begin{bmatrix}1 & -t\\ t & 1\end{bmatrix}$$

$$\text{and } (I - A) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}0 & -t\\ t & 0\end{bmatrix} = \begin{bmatrix}1 & t\\ -t & 1\end{bmatrix}$$

$$\text{Now, } \quad (I - A) \cdot \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix} = \begin{bmatrix}1 & t\\ -t & 1\end{bmatrix}\begin{bmatrix}\frac{1-t^2}{1+t^2} & \frac{-2t}{1+t^2}\\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1-t^2}{1+t^2} + \frac{2t^2}{1+t^2} & \frac{-2t}{1+t^2} + \frac{t(1-t^2)}{1+t^2}\\ \frac{-t(1-t^2)}{1+t^2} + \frac{2t}{1+t^2} & \frac{2t^2}{1+t^2} + \frac{1-t^2}{1+t^2}\end{bmatrix} = \begin{bmatrix}1 & -t\\ t & 1\end{bmatrix} = (I + A).$$

$$\text{Hence, } (I + A) = (I - A) \cdot \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}.$$

MONTHLY TEST DRIVE

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80 **Time Taken : 60 Min.**

Series 3 : Matrices and Determinants

Only One Option Correct Type

1. If $D = \text{diag}\,(d_1, d_2, d_3, ..., d_n)$ where $d_i \geq 0\ \forall\ i$, then D^{-1} equals
(a) D
(b) $\text{diag}\,(d_1^n, d_2^n, ..., d_n^n)$
(c) I_n
(d) $\text{diag}\,(d_1^{-1}, d_2^{-1}, d_3^{-1}, ... d_n^{-1})$

2. Which of the following statements is correct?
(a) A skew symmetric matrix of odd order is always invertible
(b) A skew symmetric matrix of odd order is always non-invertible.
(c) A skew symmetric matrix of even order is always non-invertible.
(d) None of these

3. If a, b, c are in G.P. or $(x - \alpha)$ is a factor of $ax^2 + 2bx + c = 0$, then value of the determinant

$$\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$$ is

(a) 0 (b) 1 (c) 2 (d) 3

4. If $A^5 = O$ such that $A^n \neq I$ for $1 \leq n \leq 4$, then $(I - A)^{-1}$ is equal to
(a) A^4 (b) A^3
(c) $I + A$ (d) None of these

5. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 3 & -1 & 9 \end{bmatrix}$, then the value of $\det(\text{adj}\,(\text{adj}\,A))$ equals
(a) 11 (b) 121
(c) 1331 (d) 14641

6. The determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$, if
(a) x, y, z are in A.P. (b) x, y, z are in G.P.
(c) x, y, z are in H.P. (d) xy, yz, zx are in A.P.

One or More Than One Option(s) Correct Type

7. A and B are square matrices with the same order. If $AB = BA$ and A and B are skew hermitian, then
(a) $A + B$ is skew hermitian
(b) AB is skew hermitian
(c) BA is hermitian
(d) All are true

8. Let $a, \lambda, \mu \in R$. Consider the system of linear equations $ax + 2y = \lambda$, $3x - 2y = \mu$. Which of the following statement(s) is(are) correct?
(a) If $a = -3$, then the system has infinitely many solutions for all values of λ and μ.
(b) If $a \neq -3$, then the system has a unique solution for all values of λ and μ.
(c) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$.
(d) If $\lambda + \mu \neq 0$, then the system has no solution for $a = -3$.

9. Value of $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$ equals
(a) 2 (b) 1 (c) 4 (d) 0

10. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ be a matrix. If $A^{10} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

(a) number of divisors of a is 11
(b) a is an odd integer
(c) $(a + b + d)$ is an even integer
(d) $a + d$ is a multiple of 13

11. If $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal, then

(a) $a = \pm\frac{1}{\sqrt{2}}$ (b) $b = \pm\frac{1}{\sqrt{6}}$

(c) $c = \pm\frac{1}{\sqrt{3}}$ (d) None of these

12. The value of θ for which the system of linear equations in x, y, z given as
$(\sin 3\theta)\, x - y + z = 0$
$(\cos 2\theta)x + 4y + 3z = 0$
$2x + 7y + 7z = 0$
has a non-trivial solution, is/are

(a) $\frac{n\pi}{2}$ (b) $\frac{2n\pi}{3}$

(c) $n\pi + (-1)^n \frac{\pi}{6}$ (d) None of these

13. If A is symmetric and B is skew-symmetric matrix, then which of the following is/are not true?
(a) ABA^T is symmetric matrix
(b) $AB^T + BA^T$ is symmetric matrix
(c) $(A + B)(A - B)$ is skew symmetric
(d) $(A + I)(B - I)$ is skew symmetric

Comprehension Type

Let α, β be the roots of the equation $ax^2 + bx + c = 0$. Also,

let $S_n = \alpha^n + \beta^n$ for $n \geq 1$ and $\Delta = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$

14. If $\Delta < 0$ then the equation $ax^2 + bx + c = 0$ has
(a) positive real roots (b) negative real roots
(c) equal roots (d) imaginary roots

15. If a, b, c are rational and one of the roots of the equation is $1+\sqrt{2}$, then the value of Δ is
(a) 8 (b) 12 (c) 30 (d) 32

Matrix Match Type

16. Match the following:

	Column-I	Column-II	
P.	If ω is a non real cube root of unity, then a root of the following equation $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$ is	1.	0
Q.	If $A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{bmatrix}$, then rank of A is	2.	2
R.	Suppose that system of equations $x = cy + bz$, $y = az + cx$, $z = bx + ay$ has non trivial solution. Then $a^2 + b^2 + c^2 + 2abc =$	3.	1
S.	$\begin{vmatrix} a+1 & a+2 & a+4 \\ a+3 & a+5 & a+8 \\ a+7 & a+10 & a+14 \end{vmatrix}$	4.	4

	P	Q	R	S
(a)	1	3	3	2
(b)	2	4	3	1
(c)	4	2	3	1
(d)	4	1	2	3

Integer Answer Type

17. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $\det(A^{2005})$ equals to

18. If $A^2 = 3A - 2I$ and $A^8 = pA + qI$, then $p + q$ is

19. If A, B and C are $n \times n$ matrices and $\det(A) = 2$, $\det(B) = 3$ and $\det(C) = 5$, then the value of $[\det(A^2BC^{-1})]$ (where $[\cdot]$ represents greatest integer function) is

20. If $M = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $M^2 - \lambda M - I_2 = O$, λ must be

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BRAIN @ WORK

Limit, Continuity and Differentiability

LEFT AND RIGHT LIMITS

Let $y = f(x)$ be a given function, and $x = a$ is the point under consideration. Left tendency of $f(x)$ at $x = a$ is called its left limit and right tendency is called its right limit.

$f(a+0) = \lim_{h\to 0} f(a+h)$ and $f(a-0) = \lim_{h\to 0} f(a-h)$,

where 'h' is a small positive number.

Thus for the existence of the limit of $f(x)$ at $x = a$, $f(a-0) = f(a+0)$

Reasons for non-existence of the limit

$\lim_{x\to a} f(x)$ will not exist due to any of these three reasons :

(a) $f(x)$ is not defined in the neighbourhood of $x = a$.

(b) Left and right limits of $f(x)$ are not the same.

(c) $f(x)$ oscillates between two fixed values as x approaching a.

SOME STANDARD LIMITS

- $\lim_{x\to 0} \frac{\sin ax}{x} = a, \ \forall \ a \in R$
- $\lim_{x\to 0} \frac{\tan ax}{x} = a, \ \forall \ a \in R$
- $\lim_{x\to 0} \frac{\sin^{-1} ax}{x} = a, \ \forall \ a \in R$
- $\lim_{x\to 0} \frac{\tan^{-1} ax}{x} = a, \ \forall \ a \in R$
- $\lim_{x\to 0} (1+ax)^{1/x} = e^a = \lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x, \forall \ a \in R$
- $\lim_{x\to 0} \frac{\log_a (1+x)}{x} = \log_a e \ \ (a>0, a\neq 1)$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a, \ \ a>0$
- $\lim_{x\to\infty} \frac{\log x}{x^m} = 0 \ \ (m>0)$
- $\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$, where m,n are rational numbers
- $\lim_{x\to 0} \frac{(1+x)^m - 1}{x} = m$

SOME THEOREMS ON LIMITS

- Let $f(x)$ and $g(x)$ are two functions such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist finitely, then

(i) $\lim_{x\to a} (f \pm g)(x)$ and $\lim_{x\to a} f(x)\,g(x)$ exist.

(ii) If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists.

However, the converse of any of the above is not necessarily true.

- $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$ provided $\lim_{x\to a} g(x)$ exists and lies in the domain of $f(x)$.

METHODS FOR CALCULATING THE LIMITS

1. When the limit is of the form $\lim_{x\to a}(f(x))^{g(x)}$ and $\lim_{x\to a}(g(x)) = \infty$

Here we discuss two different cases :

(i) When $\lim_{x\to a} f(x) = 1$.

In this case, $\lim_{x\to a} (f(x))^{g(x)} = e^{\lim_{x\to a} g(x)[f(x)-1]}$

(ii) When $\lim_{x\to a} f(x) \neq 1$, but $f(x)$ is positive in the neigbourhood of $x = a$.

In this case we write, $(f(x))^{g(x)} = e^{g(x)\ln f(x)}$

$$\Rightarrow \lim_{x\to a}(f(x))^{g(x)} = e^{\lim_{x\to a} g(x)\cdot\ln(f(x))}$$

2. L'Hospital's Rule for Calculating Limits

Let f and g be two differentiable functions in the neighbourhood of a which satisfy

(i) $g'(x) \neq 0$ for any x in the neighbourhood of a

(ii) $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$.

Then, $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

This is L' Hospital's Rule for $\frac{0}{0}$ form. However if $\lim_{x\to a} f'(x) = 0$ and $\lim_{x\to a} g'(x) = 0$, then again we can apply the L' Hospital's rule.

3. Sandwich Theorem

Let f, g and h be three functions such that in the neighbourhood of a, $f(x) \le g(x) \le h(x)$ and $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$.

CONTINUITY OF A FUNCTION

A function $f(x)$ is said to be continuous at $x = a$ if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

i.e. L.H.L.=R.H.L. = Value of the function at a or

$$\lim_{x\to a} f(x) = f(a)$$

Note : $f(x)$ will be discontinuous at $x = a$ in any of the following cases:

(i) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but are not equal.

(ii) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal but not equal to $f(a)$.

(iii) $f(a)$ is not defined.

(iv) At least one of the limits does not exist.

Properties of a Continuous Function

- A function $f(x)$, continuous on the interval $[a, b]$ possesses the following properties:

(i) $f(x)$ is bounded on $[a, b]$, *i.e.* there exist m and M such that $m \le f(x) \le M$ for all $x \in [a, b]$. Here m and M are minimum and maximum values of $f(x)$ respectively.

(ii) Let A be any number such that $\min f(x) \le A \le \max f(x)$, then there exists a point $x_0 \in [a, b]$ such that $f(x_0) = A$. It is called intermediate value theorem.

- If f and g are continuous at $x = a$, then $f + g$, kf and fg are continuous at $x = a$. Moreover, if $g(a) \neq 0$, then f/g is also continuous at $x = a$. Further if g is continuous at a and f is continuous at $g(a)$, then the composition fog is continuous at $x = a$.

DIFFERENTIABILITY

Let $y = f(x)$ be a continuous function at a point $x = a$. It is said to be differentiable at $x = a$ if $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists and is finite.

If this limit exists, we call it the derivative of $y = f(x)$ at a and is denoted by $f'(a)$.

RIGHT AND LEFT HAND DERIVATIVES

- Right hand derivative of $f(x)$ at $x = a$ is given by

$$Rf'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

- Left hand derivative of $f(x)$ at $x = a$ is given by

$$Lf'(a) = \lim_{h\to 0}\frac{f(a-h) - f(a)}{-h}$$

Clearly, $f(x)$ is differentiable at $x = a$ if and only if $Rf'(a) = Lf'(a)$.

Note : (i) If a function $f(x)$ is differentiable at $x = a$ then, it is also continuous at $x = a$. But if a function is continuous at a point, it is not necessarily differentiable at that point.

(ii) Geometrically, we interpret $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ as the slope of the graph at the point $(a, f(a))$. The line through $(a, f(a))$ which has this slope is called the tangent line at $(a, f(a))$. Thus, if there is no tangent line at a certain point, the function is not differentiable at that point. In other words, a function is not differentiable at a point where the curve suddenly changes direction (corner point) or the tangent is vertical at some point.

PROBLEMS

1. The function $f(x) = \begin{cases} 2x+1, & x \in Q \\ (x-1)^2 + 4, & x \notin Q \end{cases}$ is

(a) continuous nowhere

(b) differentiable nowhere

(c) continuous but not differentiable exactly at one point

(d) differentiable and continuous only at one point and discontinuous elsewhere

2. The value of $\lim_{x\to\frac{\pi}{2}}\frac{\int_{\pi/2}^{x} t\,dt}{\sin(2x-\pi)}$ is

(a) ∞ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

3. The value of p for which the function

$$f(x)=\begin{cases}\dfrac{(4^x-1)^3}{\sin\frac{x}{p}\log\left[1+\frac{x^2}{3}\right]}, & x\neq 0\\ 12(\log 4)^3, & x=0\end{cases}$$

may be continuous at $x = 0$, is

(a) 1 (b) 2
(c) 3 (d) none of these

4. ABC is an isosceles triangle inscribed in a circle of radius r . If $AB = AC$ and h is the altitude from A to BC and p be the perimeter of ABC then $\lim\limits_{h\to 0}\frac{\Delta}{p^3}$ equals (where Δ is the area of the triangle)

(a) $\frac{1}{32r}$ (b) $\frac{1}{64r}$
(c) $\frac{1}{128r}$ (d) none of these

5. The function $f(x) = [x]^2 - [x^2]$, (where $[\cdot]$ is the greatest integer function), is discontinuous at
(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0
(d) all integers except 1

6. If $f(x) = a|\sin x| + be^{|x|} + c|x|^3$ and if $f(x)$ is differentiable at $x = 0$, then
(a) $b = 0, c = 0, a$ is any real
(b) $a = 0, b = 0, c$ is any real
(c) $c = 0, a = 0, b$ is any real
(d) $a + b = 0$

7. Let $f : R \to R$ be a function defined by $f(x) = \min\{x + 1, |x| + 1\}$. Which of the following is true?
(a) $f(x) \geq 1$ for all $x \in R$.
(b) $f(x)$ is not differentiable at $x = 1$.
(c) $f(x)$ is differentiable everywhere.
(d) $f(x)$ is not differentiable at $x = 0$.

8. Let α and β be the roots of the equation, $ax^2 + bx + c = 0$ where $1 < \alpha < \beta$. If $\lim\limits_{x\to m}\frac{|ax^2+bx+c|}{ax^2+bx+c}=1$, then
(a) $a > 0$ and $m > 1$
(b) $a < 0$ and $m < 1$
(c) $a < 0$ and $\alpha < m < \beta$
(d) $\frac{|a|}{a}=1$ and $m > \alpha$

9. The function $f : R - \{0\} \to R$, $f(x)=\frac{1}{x}-\frac{2}{e^{2x}-1}$ can be made continuous at $x = 0$ by defining $f(0)$ as
(a) 2 (b) –1 (c) 0 (d) 1

10. If $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all x and $f(2) = 4 = f'(2)$. Then the value of $f^2(4) + g^2(4)$ is
(a) 8 (b) 16 (c) 32 (d) 64

11. The number of values of a for the which the function $f(x) = (x + 1)|x - a|$ is differentiable $\forall\, x \in R$, is
(a) 0 (b) 1
(c) 2 (d) more than 2

12. If $\lim\limits_{x\to\infty}\left(\frac{x^3+1}{x^2+1}-(ax+b)\right)=2$, then
(a) $a = 1, b = 1$ (b) $a = 1, b = 2$
(c) $a = 1, b = -2$ (d) none of these

13. $\lim\limits_{x\to\infty}\left(\frac{1}{e}-\frac{x}{1+x}\right)^x$ is equal to
(a) $\frac{e}{1-e}$ (b) 0
(c) $\frac{e}{e^{1-e}}$ (d) does not exist

14. $\lim\limits_{x\to\infty}\left(\frac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{2x-1}}$ is equal to
(a) 0 (b) ∞
(c) $\frac{1}{2}$ (d) none of these

15. $\lim\limits_{x\to\infty}\frac{2\sqrt{x}+3\sqrt[3]{x}+4\sqrt[4]{x}+\ldots\ldots+n\sqrt[n]{x}}{\sqrt{(2x-3)}+\sqrt[3]{(2x-3)}+\ldots\ldots+\sqrt[n]{(2x-3)}}$ is equal to
(a) 1 (b) ∞
(c) $\sqrt{2}$ (d) none of these

16. If $f(x)=\lim\limits_{n\to\infty}\left(\frac{1}{x^n-1}\right)n$, then for $x, y > 0, f(xy)$ is equal to
(a) $f(x)\,f(y)$ (b) $f(x) + f(y)$
(c) $f(x) - f(y)$ (d) none of these

17. The value of $\lim_{n\to\infty}\left(\sum_{r=1}^{n}\left(\frac{2r+5}{r^2+r}\right)\left(\frac{3}{5}\right)^{r+1}\right)$ is equal to

(a) 0 (b) $\frac{9}{5}$

(c) $\frac{3}{5}$ (d) none of these

18. $\lim_{x\to 0}\left\{(1+x)^{\frac{2}{x}}\right\}$ (where $\{\cdot\}$ denotes the fractional part of x) is equal to

(a) e^2-7 (b) e^2-8

(c) e^2-6 (d) none of these

19. $\lim_{x\to 0}\frac{\cos(\tan x)-\cos x}{x^4}$ is equal to

(a) $\frac{1}{6}$ (b) $-\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 1

20. Among (i) $\lim_{x\to\infty}\sec^{-1}\left(\frac{x}{\sin x}\right)$

and (ii) $\lim_{x\to\infty}\sec^{-1}\left(\frac{\sin x}{x}\right)$

(a) (i) exists, (ii) does not exist

(b) (i) does not exist, (ii) exists

(c) both (i) and (ii) exist

(d) neither (i) nor (ii) exists

21. $\lim_{x\to 1}\frac{(1-x)(1-x^2).....(1-x^{2n})}{\{(1-x)(1-x^2)....(1-x^n)\}^2}, n\in N =$

(a) ${}^{2n}P_n$ (b) ${}^{2n}C_n$

(c) $2n!$ (d) none of these

22. If $\lim_{x\to 0}(x^{-3}\sin 3x+ax^{-2}+b)$ exists and is equal to 0, then

(a) $a=-3$ and $b=\frac{9}{2}$ (b) $a=3$ and $b=\frac{9}{2}$

(c) $a=-3$ and $b=\frac{-9}{2}$ (d) $a=3$ and $b=\frac{-9}{2}$

23. If $f(x)=\begin{cases}3(1+|\tan x|)^{\frac{\alpha}{|\tan x|}}; & \frac{-1}{2}<x<0\\ \beta & ;x=0\\ 3\left(1+\left|\frac{\sin x}{3}\right|\right)^{\frac{6}{|\sin x|}}; & 0<x<\frac{2}{3}\end{cases}$

is a continuous function at $x=0$, then the ordered pair (α, β) is equal to

(a) $(2, e^2)$ (b) $\left(2, \frac{2}{e^2}\right)$

(c) $(2, 3e^2)$ (d) $\left(2, \frac{3}{e^2}\right)$

24. Let $f:(1,\infty)\to(0,\infty)$ be a continuous decreasing function with $\lim_{x\to\infty}\frac{f(4x)}{f(8x)}=1$. Then $\lim_{x\to\infty}\frac{f(6x)}{f(8x)}$ is equal to

(a) $\frac{4}{8}$ (b) $\frac{4}{6}$ (c) $\frac{6}{8}$ (d) 1

25. The value of $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m$ is

(a) 1 (b) e

(c) e^{-1} (d) none of these

26. Given $f(x)=\begin{cases}x & \text{for}|x|\le 1\\ 1 & \text{for}|x|>1\end{cases}$,

$g(x)=\begin{cases}\cos\left(\frac{\pi}{2}x\right) & \text{for}\,|x|\le 1\\ |x-1| & \text{for}\,|x|>1\end{cases}$,

$h(x)=\begin{cases}\frac{(|x|-1)}{\log_a|x|} & \text{for}\,|x|\ne 1\\ \ln a & \text{for}\,|x|=1, a>0, a\ne 1\end{cases}$

If l, m, n denote the number of points of discontinuity of the functions f, g and h in their domain respectively, over R, then (l, m, n) is

(a) (0, 0, 0) (b) (1, 1, 1)

(c) (2, 2, 2) (d) (1, 1, 0)

27. $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x+....+n^x}{n}\right)^{\frac{1}{x}}$ is equal to

(a) $(n!)^n$ (b) $(n!)^{1/n}$

(c) $n!$ (d) $\ln(n!)$

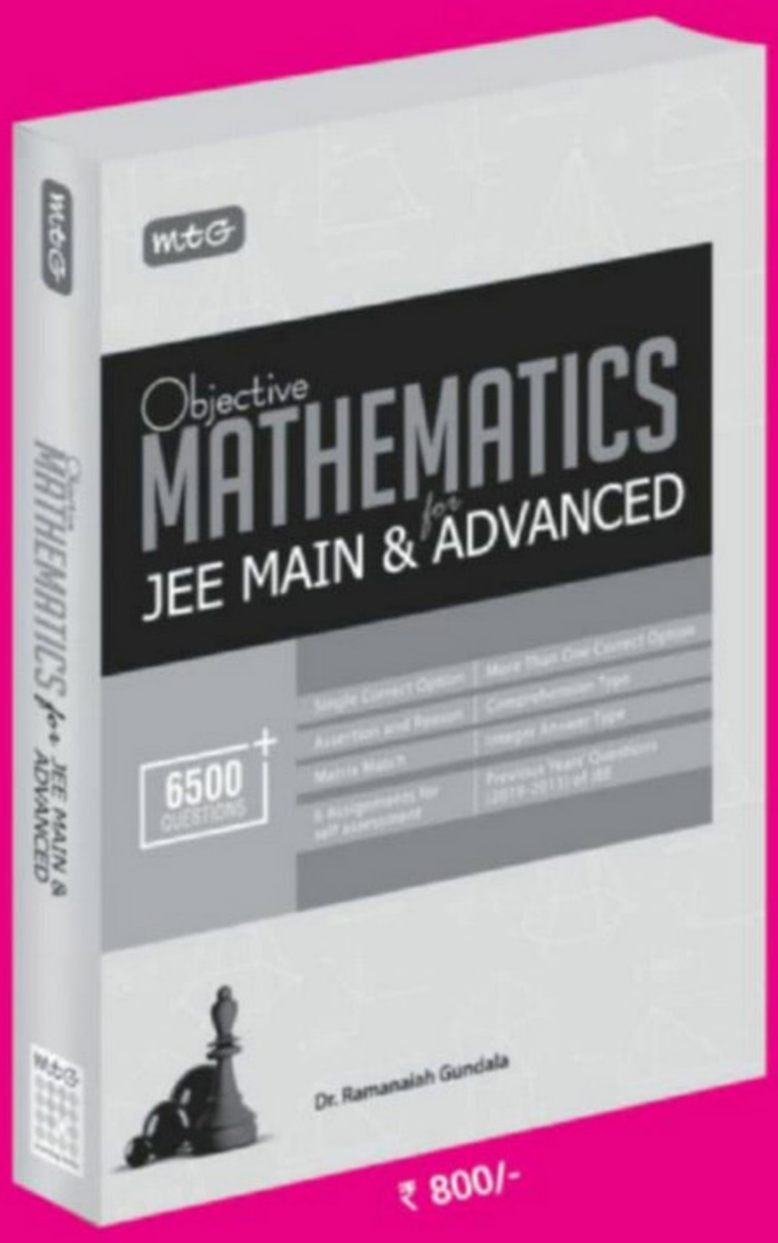
28. Let $f(x)=\frac{1}{1+e^{\frac{1}{x}}}$ for $x\ne 0$ and $f(0)=0$, then

(a) $f(0^+)$ does not exist (b) $f(0^-)=0$

(c) $f'(0^-)=1$ (d) $f'(0^+)=0$

29. Let $f(x)=\lim_{n\to\infty}\frac{1}{\left(\frac{3}{\pi}\tan^{-1}2x\right)^{2n}+5}$. Then the set of values of x for which $f(x)=0$ is

ABRACADABRA



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(a) $|2x| > \sqrt{3}$ (b) $|2x| < \sqrt{3}$

(c) $|2x| \geq \sqrt{3}$ (d) $|2x| \leq \sqrt{3}$

30. If $f(x) = \begin{cases} x^n \sin\left(\frac{1}{x^2}\right) & , x \neq 0, n \in I \\ 0 & , x = 0 \end{cases}$

Then,

(a) $\lim\limits_{x \to 0} f(x)$ exists for $n > 1$

(b) $\lim\limits_{x \to 0} f(x)$ exists for $n < 0$

(c) $\lim\limits_{x \to 0} f(x)$ does not exists for any value of n

(d) $\lim\limits_{x \to 0} f(x)$ cannot be determined

SOLUTIONS

1. (d) : $y = (x-1)^2 + 4 = x^2 - 2x + 5$

According to question,

$y = x^2 - 2x + 5 = 2x + 1$

$\Rightarrow \quad x^2 - 4x + 4 = 0 = (x-2)^2 = 0 \quad \Rightarrow \quad x = 2$

$\therefore$ $f(x)$ is continuous and differentiable at $x = 2$ only.

2. (c) : Let $y = \lim\limits_{x \to \pi/2} \dfrac{\int_{\pi/2}^{x} t\,dt}{\sin(2x-\pi)} \Rightarrow y = \lim\limits_{x \to \pi/2} \dfrac{\left[\frac{t^2}{2}\right]_{\pi/2}^{x}}{\sin(2x-\pi)}$

$$\therefore y = \lim_{x \to \frac{\pi}{2}} \frac{\left(\frac{x^2}{2} - \frac{\pi^2}{8}\right)}{\sin(2x-\pi)} \Rightarrow y = \lim_{x \to \frac{\pi}{2}} \frac{(4x^2 - \pi^2)}{8[\sin(2x-\pi)]}$$

$$\Rightarrow \quad y = \lim_{x \to \pi/2} \frac{1}{8} \frac{(2x-\pi)(2x+\pi)}{\sin(2x-\pi)}$$

$$\therefore \quad y = \frac{1}{8} \frac{\lim\limits_{x \to \pi/2} (2x+\pi)}{\lim\limits_{x \to \pi/2} \frac{\sin(2x-\pi)}{(2x-\pi)}} \quad \Rightarrow \quad y = \frac{1}{8} \times 2\pi = \frac{\pi}{4}$$

3. (d) : For $f(x)$ to be continuous at $x = 0$, we have

$$\lim_{x \to 0} f(x) = f(0) = 12(\log 4)^3 \qquad \ldots(i)$$

$$\text{Now, } \lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{4^x - 1}{x}\right)^3 \times \frac{\left(\frac{x}{p}\right)}{\left(\sin\frac{x}{p}\right)} \cdot \frac{px^2}{\log\left(1 + \frac{1}{3}x^2\right)}$$

$$= (\log 4)^3 \cdot 1 \cdot p \cdot \lim_{x \to 0} \left(\frac{x^2}{\frac{1}{3}x^2 - \frac{1}{18}x^4 + \ldots}\right) = 3p\,(\log 4)^3$$

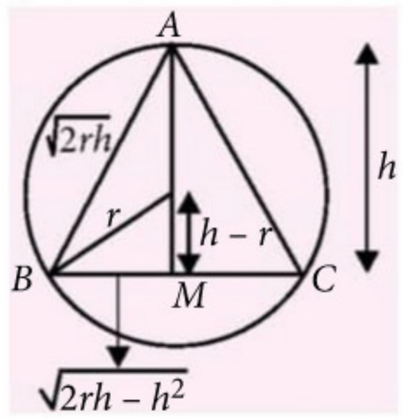
$\therefore$ On comparing with (i), we get $p = 4$.

4. (c) : We have, $AB = \sqrt{h^2 + 2rh - h^2} = \sqrt{2rh}$

$$p = 2\sqrt{2rh} + 2\sqrt{2rh - h^2} = 2[\sqrt{2rh} + \sqrt{2rh - h^2}]$$

$$\text{Now, } \Delta = \frac{1}{2} 2h\sqrt{2rh - h^2} = h\sqrt{2rh - h^2}$$

$$\therefore \lim_{h \to 0} \frac{\Delta}{p^3}$$



$$= \lim_{h \to 0} \frac{h\sqrt{2rh - h^2}}{8[\sqrt{2rh} + \sqrt{2rh - h^2}]^3}$$

$$= \lim_{h \to 0} \frac{h^{3/2}\sqrt{2r - h}}{8h^{3/2}[\sqrt{2r} + \sqrt{2r - h}]^3}$$

$$= \frac{\sqrt{2r}}{8 \cdot 8(2r)(2r)^{1/2}} = \frac{1}{128r}$$

5. (d) : Given $f(x) = [x]^2 - [x^2]$

Let $x = n$, $n \in I$

Now, $\lim\limits_{x \to n^+} f(x) = n^2 - n^2 = 0$

and $\lim\limits_{x \to n^-} f(x) = (n-1)^2 - (n^2 - 1) = -2n + 2$

Also, $f(n) = 0$

For the function to be continuous at $x = n$

$-2n + 2 = 0 \Rightarrow \quad n = 1$

Hence, function is discontinuous at all integers except 1.

6. (d) : $f'(0^-) = \lim\limits_{h \to 0} \dfrac{a\sin h + be^h + ch^3 - b}{-h} = -(a+b)$

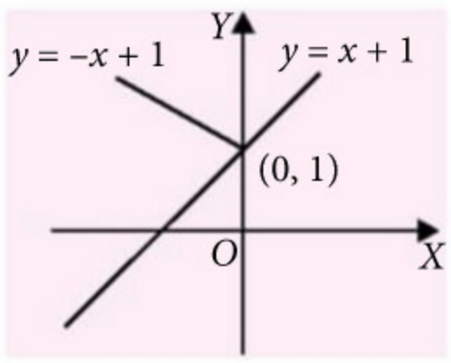
$$f'(0^+) = \lim_{h \to 0} \frac{a\sin h + be^h + ch^3 - b}{h}$$

$$= \frac{a(\sin h)}{h} + \frac{b(e^h - 1)}{h} + \frac{ch^3}{h} = a + b$$

$\because$ $f(x)$ is differentiable at $x = 0$

$\therefore$ $a + b = -(a+b) \Rightarrow (a+b) = 0.$

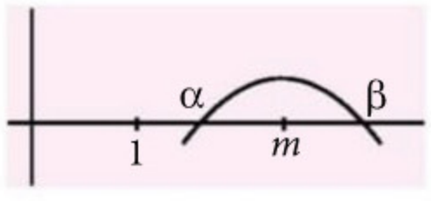
7. (c) : Given, $f(x) = \min\{x + 1, |x| + 1\}$



$\therefore$ $f(x) = x + 1 \ \forall\ x \in R$

8. (c) : Given, $\lim\limits_{x\to m}\dfrac{|ax^2+bx+c|}{ax^2+bx+c}=1$

$=\lim\limits_{x\to m}\dfrac{|a(x-\alpha)(x-\beta)|}{a(x-\alpha)(x-\beta)}$



If $a<0$ and $\alpha<m<\beta$, then

$a\,(m-\alpha)\,(m-\beta)>0$

$\Rightarrow$ Limit is 1.

9. (d) : $\lim\limits_{x\to0}\left(\dfrac{1}{x}-\dfrac{2}{e^{2x}-1}\right)=\lim\limits_{x\to0}\dfrac{e^{2x}-1-2x}{x(e^{2x}-1)}$

$$=\lim_{x\to0}\frac{\left\{\left(1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\ldots\ldots\right)-1-2x\right\}}{x\left\{\left(1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\ldots\ldots\right)-1\right\}}=1=f(0)$$

10. (c) : Given, $f'(x)=g(x)$

$\therefore\ f''(x)=g'(x)=-f(x)\quad\Rightarrow f''(x)+f(x)=0$

Solution of the differential equation is given by

$f(x)=c_1\cos x+c_2\sin x$ ...(i)

$\Rightarrow\ f'(x)=-c_1\sin x+c_2\cos x=g(x)$ ...(ii)

Now, $f(2)=4\Rightarrow c_1\cos 2+c_2\sin 2=4$ ...(iii)

$f'(2)=4\Rightarrow -c_1\sin 2+c_2\cos 2=4$ ...(iv)

Squaring and adding, (iii) and (iv), we get

$c_1^2+c_2^2=32$ ...(v)

Now, $f(4)=c_1\cos4+c_2\sin4$ ...(vi)

$g(4)=-c_1\sin4+c_2\cos4$ ...(vii)

Squaring and adding, (vi) and (vii), we get

$f^2(4)+g^2(4)=c_1^2+c_2^2=32$ ...(viii)

$\therefore\ f^2(4)+g^2(4)=32$

11. (b) : If function is differentiable in R, then it must be differentiable at $x_0=a$.

$$f'(a^+)=\lim_{h\to0}\frac{(a+h+1)|h|-0}{h}=a+1$$

$$f'(a^-)=\lim_{h\to0}\frac{(a-h+1)|-h|}{-h}=-(a+1)$$

$\therefore\ f'(a^+)=f'(a^-)\Rightarrow a=-1$

12. (c) : Given, $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2$

$$\Rightarrow\quad\lim_{x\to\infty}\frac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1}=2$$

$\Rightarrow\ 1-a=0$ and $-b=2\ \Rightarrow\ a=1,\ b=-2.$

13. (d) : $\lim\limits_{x\to\infty}\left(\dfrac{1}{e}-\dfrac{x}{1+x}\right)^x=\lim\limits_{x\to\infty}\left(\dfrac{\frac{1}{e}-\frac{1}{\frac{1}{x}+1}}{}\right)^x$

$=\left(\dfrac{1}{e}-1\right)^\infty=$ (some negative value)$^\infty$,

which is not defined.

14. (c) : $\lim\limits_{x\to\infty}\left(\dfrac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{2x-1}}$

$$=\lim_{x\to\infty}\left(\frac{1+\frac{2}{x}-\frac{1}{x^2}}{2-\frac{3}{x}-\frac{2}{x^2}}\right)^{\frac{2+1/x}{2-1/x}}=\frac{1}{2}$$

15. (c) : Since highest degree of x is $\dfrac{1}{2}$, divide numerator and denominator by $\sqrt{x}$, then value of limit becomes $\dfrac{2}{\sqrt{2}}$ or $\sqrt{2}$.

16. (b) : $\lim\limits_{n\to\infty}n\,(x^{1/n}-1)=\lim\limits_{n\to\infty}\dfrac{x^{1/n}-1}{\frac{1}{n}}$

$=\lim\limits_{m\to0}\dfrac{x^m-1}{m}\left(\text{where }\dfrac{1}{n}\text{ is replaced by }m\right)=\ln x$

Now, $f(xy)=\ln(xy)=\ln x+\ln y=f(x)+f(y)$.

17. (b) : Let $T_r=\left(\dfrac{2r+5}{r(r+1)}\right)\left(\dfrac{3}{5}\right)^{r+1}$

$$=\left(\frac{5(r+1)-3r}{r(r+1)}\right)\left(\frac{3}{5}\right)^{r+1}=3\left[\frac{1}{r}\left(\frac{3}{5}\right)^r-\frac{1}{r+1}\left(\frac{3}{5}\right)^{r+1}\right]$$

$=3(v_r-v_{r+1})$

So, $S_n=\sum\limits_{r=1}^{n}T_r=3\left[\sum\limits_{r=1}^{n}v_r-\sum\limits_{r=1}^{n}v_{r+1}\right]$

$$\Rightarrow\quad S_n=3(v_1-v_{n+1})=\frac{9(n+1)5^n-3^{n+2}}{(n+1)\,5^{n+1}}$$

So, $\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\dfrac{9(n+1)5^n-3^{n+2}}{(n+1)\,5^{n+1}}=\dfrac{9}{5}$.

18. (a) : $\{(1+x)^{2/x}\}=(1+x)^{2/x}-[(1+x)^{2/x}]$

Now, $\lim\limits_{x\to0}(1+x)^{2/x}=e^2$

$\therefore\ \lim\limits_{x\to0}\{(1+x)^{2/x}\}=e^2-[e^2]=e^2-7.$

19. (b) : Since, $\cos(\tan x) - \cos x$

$$= 2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)$$

$$\therefore \lim_{x\to 0}\frac{\cos(\tan x)-\cos x}{x^4}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)}{x^4}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)}{x^4\left(\frac{x+\tan x}{2}\right)\left(\frac{x-\tan x}{2}\right)}\left(\frac{x^2-\tan^2 x}{4}\right)$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{x^2-\tan^2 x}{x^4}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{x^2-\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+.....\right)^2}{x^4}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{1}{x^2}\left(1-\left(1+\frac{x^2}{3}+\frac{2}{15}x^4+.....\right)^2\right) = -\frac{1}{3}$$

20. (a) : (i) $\lim\limits_{x\to\infty}\sec^{-1}\left(\frac{x}{\sin x}\right) = \sec^{-1}\left(\frac{\infty}{\sin\infty}\right)$

$$= \sec^{-1}\left(\frac{\infty}{\text{any value between } -1 \text{ to } 1}\right)$$

$$= \sec^{-1}(\pm\infty) = \frac{\pi}{2}$$

(ii) $\lim\limits_{x\to\infty}\sec^{-1}\left(\frac{\sin x}{x}\right) = \sec^{-1}\left(\frac{\sin\infty}{\infty}\right)$

$$= \sec^{-1}\left(\frac{\text{any value between } -1 \text{ to } 1}{\infty}\right)$$

$= \sec^{-1}0 =$ not defined

Hence (i) exists but (ii) does not exist.

21. (b) : $\lim\limits_{x\to 1}\frac{(1-x)(1-x^2).....(1-x^{2n})}{\{(1-x)(1-x^2)....(1-x^n)\}^2}$

$$= \lim_{x\to 1}\frac{\left(\frac{1-x}{1-x}\right)\left(\frac{1-x^2}{1-x}\right)...\left(\frac{1-x^{2n}}{1-x}\right)}{\left(\left(\frac{1-x}{1-x}\right)\left(\frac{1-x^2}{1-x}\right)....\left(\frac{1-x^n}{1-x}\right)\right)^2}$$

$$= \frac{1\times 2\times 3.....(2n)}{(1\times 2\times 3\times....\times n)^2} = \frac{(2n)!}{n!n!} = {}^{2n}C_n$$

22. (a) : We have,

$$\lim_{x\to 0}\frac{\sin 3x}{x^3}+\frac{a}{x^2}+b = \lim_{x\to 0}\frac{\sin 3x+ax+bx^3}{x^3}$$

$$= \lim_{x\to 0}\frac{3\frac{\sin 3x}{3x}+a+bx^2}{x^2}$$

For existence of limit, $3 + a = 0 \Rightarrow a = -3$.

Now, $\lim\limits_{x\to 0}\frac{\sin 3x-3x+bx^3}{x^3} = 0$

$$= 27\lim_{t\to 0}\frac{\sin t-t}{t^3}+b = 0 \qquad (\text{Put } 3x = t)$$

$$= \frac{-27}{6}+b = 0 \Rightarrow b = \frac{9}{2}.$$

23. (c) : We have,

$$f(x) = \begin{cases} 3(1+|\tan x|)^{\frac{\alpha}{|\tan x|}} & ; \ \frac{-1}{2} < x < 0 \\ \beta & ; \ x = 0 \\ 3\left(1+\left|\frac{\sin x}{3}\right|\right)^{\frac{6}{|\sin x|}} & ; \ 0 < x < \frac{2}{3} \end{cases}$$

Now, R.H.L $= \lim\limits_{x\to 0^+} f(x)$

$$= \lim_{x\to 0^+} 3\left(1+\frac{|\sin x|}{3}\right)^{\frac{6}{|\sin x|}} = 3e^2 \qquad ...(i)$$

Similarly, L.H.L $= \lim\limits_{x\to 0^-} f(x) = 3e^{\alpha}$ ...(ii)

$\because$ $f(x)$ is continuous at $x = 0$

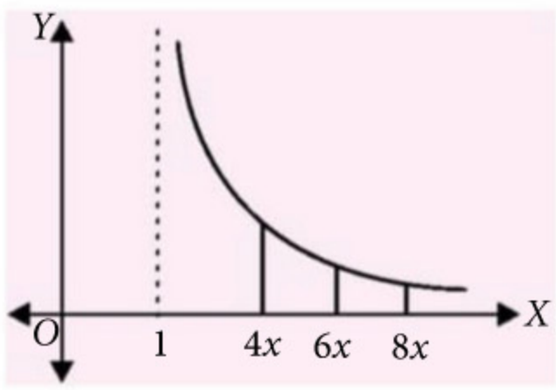
$\therefore$ $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) = f(0)$

$\Rightarrow$ $3e^{\alpha} = 3e^2 = \beta$.

So, $\alpha = 2$, $\beta = 3e^2$.

24. (d) : As $f(x)$ is a continuous decreasing positive function on $(1, \infty)$, so $0 < f(8x) < f(6x) < f(4x)$

$$\Rightarrow 0 < 1 < \frac{f(6x)}{f(8x)} < \frac{f(4x)}{f(8x)}$$



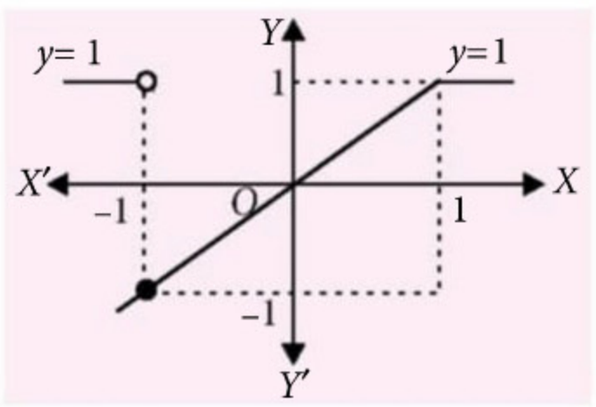
$$\Rightarrow \lim_{x\to\infty} 1 \le \lim_{x\to\infty} \frac{f(6x)}{f(8x)} \le \lim_{x\to\infty} \frac{f(4x)}{f(8x)}$$

$\therefore$ By squeeze theorem, $\lim\limits_{x\to\infty} \dfrac{f(6x)}{f(8x)} = 1$

25. (a) : $\lim\limits_{m\to\infty} \left(\cos\dfrac{x}{m}\right)^m = \lim\limits_{m\to\infty} \left[1-\left(1-\cos\dfrac{x}{m}\right)\right]^m$

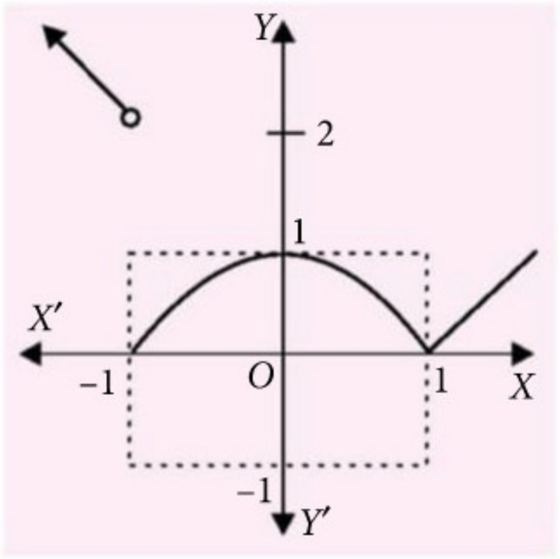
$$= \lim_{m\to\infty} \left[1-2\sin^2\frac{x}{2m}\right]^m = e^{\lim\limits_{m\to\infty}\left(-2\sin^2\frac{x}{2m}\right)m} = 1.$$

26. (d) : $f(x) = \begin{cases} x \text{ for } -1 \le x \le 1 \\ 1 \text{ for } x > 1 \text{ or } x < -1 \end{cases}$



Clearly, $f(x)$ is discontinuous at $x = -1$

$$g(x) = \begin{cases} \cos\dfrac{\pi x}{2} & \text{for } -1 \le x \le 1 \\ x-1 & \text{for } x > 1 \\ 1-x & \text{for } x < -1 \end{cases}$$



Clearly $g(x)$ is discontinuous at $x = -1$.

$$\text{Now, } h(x) = \begin{cases} \dfrac{x-1}{\log_a x}, & x > 0,\ x \ne 1 \\ \dfrac{-x-1}{\log_a(-x)}, & x < 0,\ x \ne 1 \\ \ln a, & |x| = 1,\ a \ne 1 \end{cases}$$

Clearly, $h(x)$ is continuous for all x including $x = 1$ or -1.

$\because$ $x = 0$ is not in the domain

$\therefore$ There is no point of discontinuity.

27. (b) : We have, $\lim\limits_{x\to 0} \left(\dfrac{1^x+2^x+3^x+....+n^x}{n}\right)^{\frac{1}{x}}$

$$= e^{\lim\limits_{x\to 0}\left(\frac{1^x-1}{n}+\frac{2^x-1}{n}+....+\frac{n^x-1}{n}\right)\frac{1}{x}}$$

$$= e^{\lim\limits_{x\to 0}\frac{1}{n}\left\{\frac{1^x-1}{x}+\frac{2^x-1}{x}+....+\frac{n^x-1}{x}\right\}}$$

$$= e^{\frac{1}{n}[\log 1+\log 2+....+\log n]} = e^{\frac{1}{n}(\log n!)} = e^{(\log n!)^{1/n}} = (n!)^{1/n}$$

28. (d) : $f(x) = \begin{cases} \dfrac{1}{1+e^{1/x}}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$$\therefore \quad f'(0^+) = \lim_{h\to 0} \frac{\frac{1}{1+e^{1/h}}}{h} = \lim_{h\to 0} \frac{1/h}{1+e^{1/h}}$$

$$= \lim_{h\to 0} \frac{-1/h^2}{e^{1/h}(-1/h^2)} = 0$$

Now, $f'(0^-) = \lim\limits_{h\to 0} \dfrac{\frac{1}{1+e^{-1/h}}}{-h} \to -\infty;$

Also, $f(0^+) = \lim\limits_{h\to 0} \dfrac{1}{1+e^{1/h}} = 0;\ f(0^-) = \lim\limits_{h\to 0} \dfrac{1}{1+e^{-1/h}} = 1$

29. (a) : Given $f(x) = \lim\limits_{n\to\infty} \dfrac{1}{\left(\frac{3}{\pi}\tan^{-1}2x\right)^{2n}+5} = 0$

$$\Rightarrow \lim_{n\to\infty} \left[\left(\frac{3}{\pi}\tan^{-1}2x\right)^2\right]^n \to \infty \Rightarrow \left(\frac{3}{\pi}\tan^{-1}2x\right)^2 > 1$$

$$\Rightarrow \left|\tan^{-1}2x\right| > \frac{\pi}{3}$$

$$\Rightarrow \tan^{-1}2x < -\frac{\pi}{3} \text{ or } \tan^{-1}2x > \frac{\pi}{3}$$

$$\Rightarrow \quad 2x < -\sqrt{3} \text{ or } 2x > \sqrt{3} \Rightarrow |2x| > \sqrt{3}$$

30. (a) : For $n > 1$,

$\lim\limits_{x\to 0} x^n \sin\left(\dfrac{1}{x^2}\right) = 0 \times (\text{any value between } -1 \text{ to } 1) = 0$

For $n < 0$,

$\lim\limits_{x\to 0} x^n \sin\left(\dfrac{1}{x^2}\right) = \infty \times (\text{any value between } -1 \text{ to } 1) = \infty$

Challenging PROBLEMS for Entrance Exams

1. For $n \in N$, the product

$$\left(4-\frac{2}{1}\right)\left(4-\frac{2}{2}\right)\left(4-\frac{2}{3}\right)\left(4-\frac{2}{4}\right)\ldots\left(4-\frac{2}{n}\right) \text{ is}$$

(a) an integer for all n
(b) a rational number for all n
(c) an integer only for $n \le 16$
(d) an irrational number

2. Let $f(n)=\sum_{k=1}^{n} k^2 \cdot ({}^nC_k)^2$, then $f(5)=$

(a) 1000 (b) 1250
(c) 1750 (d) 2500

3. If $\sum_{r=0}^{n}\left(\frac{r^3+2r^2+3r+2}{(r+1)^2}\right){}^nC_r=\frac{2^2+2^3+2^4-2^1}{3}$,

then $n=$

(a) 2 (b) 4 (c) 8 (d) 16

4. Let $T=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$ and

$Y={}^nC_1-\frac{1}{2}\cdot{}^nC_2+\frac{1}{3}\cdot{}^nC_3$ .... (n terms), then

(a) $T=\frac{1}{2}Y$ (b) $T=Y$
(c) $T=2Y$ (d) $T=\frac{Y}{3}$

5. $\sum_{k=0}^{n}\frac{{}^nC_k}{(k+1)(k+2)}=$

(a) $\frac{2^{n+2}-(n+3)}{(n+1)(n+2)}$ (b) $\frac{2^{n+3}-(n+2)}{(n+1)(n+2)}$

(c) $\frac{2^n-(n+2)}{n(n+1)}$ (d) $\frac{2^n-(n+3)}{n(n+2)}$

6. Let n be an odd integer greater than or equal to 5, then ${}^nC_1-5\cdot{}^nC_2+5^2\cdot{}^nC_3-\ldots+5^{n-1}\,{}^nC_n$ is

(a) a prime number (b) a composite number
(c) prime or composite depends on value of n
(d) a prime number only when $n>25$

7. Let n be an odd positive integer. Consider the set

$$S=\left\{{}^nC_1, {}^nC_2, \ldots\ldots, {}^nC_{\frac{n-1}{2}}\right\}, \text{ then}$$

(a) S contains an odd number of odd terms
(b) S contains an even number of odd terms
(c) S contains an odd number of even terms
(d) S contains an even number of even terms

8. For positive integers n and k, we have

$$\frac{\left({}^nC_{n-1}\right)^6+\left({}^{n-2}C_k\right)^6+\left({}^{n+3}C_{n+1}\right)^3}{3\left({}^{n-2}C_k\right)^2\left({}^{n+3}C_2\right)}=n^2, \text{ then } n+k=$$

(a) 8 (b) 0 (c) 16 (d) 60

9. Evaluate: $17\cdot{}^{20}C_0+16\cdot{}^{20}C_1+15\cdot{}^{20}C_2+\ldots.$
$+0\cdot{}^{20}C_{17}+(-1)\cdot{}^{20}C_{18}+(-2)\cdot{}^{20}C_{19}+(-3)\cdot{}^{20}C_{20}$

(a) $7\cdot2^{20}$ (b) $7\cdot2^{19}$
(c) $14\cdot2^{20}$ (d) $14\cdot2^{19}$

10. $\sum_{i=0}^{[n/2]}{}^nC_i=$ _____, $[\cdot]$ denotes greatest integer function

(a) 2^n if n is odd (b) 2^{n+1} if n is odd
(c) $2^{n-1}+\frac{1}{2}\cdot{}^nC_{n/2}$ if n is even
(d) $2^n+\frac{1}{2}\cdot{}^nC_{n/2}$ if n is even

11. $\sum_{k=0}^{49}(-1)^k\cdot\left({}^{99}C_{2k}\right)=$

(a) 2^{49} (b) -2^{49} (c) 2^{50} (d) -2^{50}

12. All solutions of (n, k) $(2 < k < n)$ such that ${}^nC_{k-1}$, ${}^nC_k$ and ${}^nC_{k+1}$ form an increasing A.P. is of the form $(t \geq 3)$

(a) $\left[t^2-2,\ \dfrac{t^2-t-2}{2}\right]$ (b) $\left[t^2-2,\ \dfrac{t^2-t}{2}\right]$

(c) $[t^2 - 2,\ t^2]$ (d) $\left[t^2-2,\ \dfrac{t^2-t+2}{2}\right]$

13. Let sequence, $a_n = {}^{n+4}C_4 - a_{n-2} + 2a_{n-1}$, $n \geq 2$, $a_0 = 0$, $a_1 = 5$ then $a_n =$

(a) ${}^{n+6}C_6 - n - 1$ (b) ${}^{n+6}C_6 - n - 2$

(c) ${}^{n+6}C_6 - n + 1$ (d) ${}^{n+6}C_6 - n + 2$

14. For a non-negative integer k, define

$T_k(n) = 1^k + 2^k + ... + n^k$ then $1+\sum_{k=0}^{r-1} {}^rC_k\, T_k(n) =$

(a) n^r (b) $(n + 1)^r$

(c) $n^r + 1$ (d) $n^r - r$

15. Let m and n be integers such that $1 \leq m \leq n$, then the number $n\sum_{k=0}^{m-1} (-1)^k \cdot {}^nC_k$ is divisible by

(a) m (b) $m + 1$

(c) $m + 2$ (d) $m + 3$

16. Let $< F_n >$ be the fibonacci sequence, $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, then for any positive integer n,

$F_1({}^nC_1) + F_2({}^nC_2) + ... + F_n({}^nC_n) =$

(a) F_{2n} (b) F_{2n+1} (c) F_{2n+2} (d) F_{2n-1}

Paragraph for Q. 17 to Q. 19

For a positive integer n, define the integers a_n, b_n and c_n by $a_n + b_n \cdot \sqrt[3]{2} + c_n \cdot \sqrt[3]{4} = (1 + \sqrt[3]{2} + \sqrt[3]{4})^n$

17. If $n = 3m$ form, then $2^{-n/3} \cdot \sum_{k=0}^{n} ({}^nC_k) \cdot a_k =$

(a) a_n (b) a_{n+1}

(c) $b_n \cdot \sqrt[3]{2}$ (d) c_n

18. If $n = 3m + 2$ form, then $2^{-n/3} \cdot \sum_{k=0}^{n} {}^nC_k \cdot a_k =$

(a) a_n (b) a_{n+1}

(c) $b_n \cdot \sqrt[3]{2}$ (d) $c_n \cdot \sqrt[3]{4}$

19. If $n = 3m + 1$ form, then $2^{-n/3} \cdot \sum_{k=0}^{n} {}^nC_k \cdot a_k =$

(a) a_n (b) a_{n+1}

(c) b_n (d) $c_n \cdot \sqrt[3]{4}$

SOLUTIONS

1. (a) : $\prod_{k=1}^{n}\left(4-\frac{2}{k}\right) = 2^n \cdot \frac{\prod(2k-1)}{n!}$

$= \frac{2^n \cdot \prod(2k-1)\cdot\prod(2k)}{n!\cdot\prod(2k)} = \frac{2^n\cdot(2n)!}{2^n\cdot(n!)^2} = {}^{2n}C_n = \text{integer}$

2. (c) : Since, $k \cdot {}^nC_k = n \cdot {}^{n-1}C_{k-1}$

$\therefore \quad \sum_{k=1}^{n} k^2 \cdot ({}^nC_k)^2 = n^2 \cdot \sum_{k=0}^{n-1} ({}^{n-1}C_k)^2$

$= n^2 \cdot ({}^{2n-2}C_{n-1})$

So, $f(5) = 25 \cdot {}^8C_4 = 1750$

3. (a) : Given summation $= \sum_{r=0}^{n} \frac{(r^2+r+2)\cdot {}^nC_r}{r+1}$

$= \sum_{r=0}^{n} r\,{}^nC_r + 2\sum_{r=0}^{n}\frac{{}^nC_r}{r+1} = n\cdot 2^{n-1} + 2\frac{[2^{n+1}-1]}{n+1}$

So, by comparing with expression on R.H.S., we have $n = 2$.

4. (b) : Notice that, $\frac{1-(1-x)^n}{x}$

$= {}^nC_1 - {}^nC_2 x + {}^nC_3 x^2 - .....$

Integrating on both sides from 0 to 1, we get

$\int_0^1 \frac{1-y^n}{1-y}\,dy = {}^nC_1 - {}^nC_2\cdot\frac{1}{2} + {}^nC_3\cdot\frac{1}{3}...$

$\int_0^1 (1+y+y^2+....+y^{n-1})\,dy = {}^nC_1 - \frac{1}{2}{}^nC_2 + \frac{1}{3}{}^nC_3...$

or $1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n} = {}^nC_1 - \frac{1}{2}\cdot{}^nC_2 + \frac{1}{3}{}^nC_3 - ...(n \text{ terms})$

5. (a) : Summing of the identity

${}^{n+2}C_k = \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)}\cdot{}^nC_k$

From $k = 0$ to $k = n$, gives

$S_n = \frac{1}{(n+2)(n+1)}\left(\sum_{k=0}^{n+2} {}^{n+2}C_k - {}^{n+2}C_{n+1} - {}^{n+2}C_{n+2}\right)$

So, $S_n = \frac{1}{(n+2)(n+1)}[2^{n+2} - (n+2) - 1] = \frac{2^{n+2}-(n+3)}{(n+1)(n+2)}$

6. (b) : Let $N = {}^nC_1 - 5\cdot{}^nC_2 + 5^2\cdot{}^nC_3 - .... + 5^{n-1}\cdot{}^nC_n$

then $5N = 1 + (-1 + 5)^n$

So,

$N = \frac{4^n+1}{5} = \frac{1}{5}\left[2^{n-1} + \left(2^{\frac{n-1}{2}} - 1\right)^2\right]\left[2^{n-1} + \left(2^{\frac{n-1}{2}} + 1\right)^2\right]$

Notice that for $n \geq 5$, both the factors are greater than 5 and one of the factors is divisible by 5. Hence, N is not a prime number.

7. (a) : Define $S_n = {}^nC_1 + {}^nC_2 + ... + {}^nC_{n-1}$, then

$2S_n = {}^nC_1 + {}^nC_2 + ... + {}^nC_{n-1} = 2^n - 2$

i.e. $S_n = 2^{n-1} - 1 =$ odd

$\Rightarrow$ S contains an odd number of odd terms.

8. (a) : Applying A.M.-G.M. inequality on $({}^nC_{n-1})^6$, $({}^{n-2}C_k)^6$ and $({}^{n+3}C_{n+1})^3$. We have A.M. = G.M. from the given condition in question.

So, all the given numbers should be same.

i.e. $({}^nC_{n-1})^6 = ({}^{n-2}C_k)^6 = ({}^{n+3}C_{n+1})^3$

i.e. $n^2 = ({}^{n-2}C_k)^2 = ({}^{n+3}C_2)$

Hence, $n = 6$ and $k = 2$

9. (a) : Given sum $= \sum_{k=0}^{20} (17-k) \cdot {}^{20}C_k$

$$= 17 \sum_{k=0}^{20} {}^{20}C_k - \sum_{k=0}^{20} k \cdot {}^{20}C_k$$

$= 17(2^{20}) - 20(2^{19}) = 7(2^{20})$

10. (c) : For $n = 2m + 1$ form (odd)

$S_n = {}^{2m+1}C_0 + {}^{2m+1}C_1 + ... + {}^{2m+1}C_m$

$\Rightarrow$ $2S_n = {}^{2m+1}C_0 + {}^{2m+1}C_1 + ... + {}^{2m+1}C_m$

$+ {}^{2m+1}C_{m+1} + ... + {}^{2m+1}C_{2m+1}$

$= 2^{2m+1}$ *i.e.* $S_n = 2^{n-1}$

Similarly for $n = 2m$ (even), $2S_n = 2^{2m} + {}^{2m}C_m$

i.e. $S_n = 2^{n-1} + \frac{1}{2} \cdot {}^nC_{n/2}$

11. (b) : $(1 + x)^{99} = {}^{99}C_0 + {}^{99}C_1 x + {}^{99}C_2 x^2 + ...$

$+ {}^{99}C_{99} x^{99}$

We put $x = i$ and $x = -i$ and add them to get

$$ {}^{99}C_0 - {}^{99}C_2 + {}^{99}C_4 - .... = \frac{(1+i)^{99} + (1-i)^{99}}{2} = -2^{49}$$

Noting that $(1 + i)^2 = 2i$ makes simplification easier.

12. (a) : For A.P., $2 \cdot {}^nC_k = {}^nC_{k-1} + {}^nC_{k+1}$ gives $(n - 2k)^2 = n + 2$ *i.e.* if $n - 2k = t$, then

$n = t^2 - 2$ and $k = \frac{t^2 - t - 2}{2}, t \geq 3$ as $n > 2$.

13. (a) : Let $a_n - a_{n-1} = b_n$, $b_1 = 5$ then given relation becomes $b_n - b_{n-1} = {}^{n+4}C_4$, $n \geq 2$

Putting $n = 2, 3, ...$ and adding all, we have

$$b_n - b_1 = \sum_{r=2}^{n} {}^{r+4}C_4 = {}^{n+5}C_5 - 6$$

i.e. $a_n - a_{n-1} = {}^{n+5}C_5 - 1$

Putting $n = 1, 2, 3, ...$ and adding all, we have

$$a_n = \sum_{r=1}^{n} {}^{r+5}C_5 - n = {}^{n+6}C_6 - 1 - n$$

14. (b) : $$1 + \sum_{k=0}^{r-1} {}^rC_k \cdot T_k(n) = 1 + \sum_{k=0}^{r-1} {}^rC_k \cdot \sum_{p=1}^{n} p^k$$

$$= 1 + \sum_{p=1}^{n} \sum_{k=0}^{r-1} {}^rC_k \cdot p^k = 1 + \sum_{p=1}^{n} [(p+1)^r - p^r]$$

$= (n + 1)^r$

15. (a) : We have, $n \cdot \sum_{k=0}^{m-1} (-1)^k \cdot {}^nC_k$

$$= n \cdot \sum_{k=0}^{m-1} (-1)^k \cdot \left[{}^{n-1}C_k + {}^{n-1}C_{k-1} \right]$$

$$= n \cdot \sum_{k=0}^{m-1} (-1)^k \cdot {}^{n-1}C_k - n \sum_{k=0}^{m-2} (-1)^k \cdot {}^{n-1}C_k$$

$$= n \cdot (-1)^{m-1} \cdot \left({}^{n-1}C_{m-1} \right) = m(-1)^{m-1} \cdot {}^nC_m,$$

which is divisible by m.

16. (a) : The general term in fibonacci sequence is given by

$$F_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n), \; n \geq 1 \text{ where } \alpha, \beta = \frac{1 \pm \sqrt{5}}{2}$$

So, $$\sum_{i=1}^{n} F_i \, ({}^nC_i) = \frac{1}{\sqrt{5}} \sum_{i=1}^{n} \left[{}^nC_i \, \alpha^i - {}^nC_i \, \beta^i \right]$$

$$= \frac{1}{\sqrt{5}} [(1+\alpha)^n - (1+\beta)^n] = \frac{1}{\sqrt{5}} [\alpha^{2n} - \beta^{2n}]$$

$= F_{2n}$ [$\because 1 + \alpha = \alpha^2$]

(17 - 19):

17. (a) **18. (c)** **19. (d)**

$$a_n + b_n \cdot \sqrt[3]{2} + c_n \cdot \sqrt[3]{4} = \frac{(\sqrt[3]{2})^n (1 + \sqrt[3]{2} + \sqrt[3]{4})^n}{(\sqrt[3]{2})^n}$$

$$= 2^{-n/3} [\sqrt[3]{2} + \sqrt[3]{4} + 2]^n = 2^{-n/3} [1 + (1 + \sqrt[3]{2} + \sqrt[3]{4})]^n$$

$$= 2^{-n/3} \cdot \sum_{k=0}^{n} {}^nC_k (a_k + b_k \cdot \sqrt[3]{2} + c_k \cdot \sqrt[3]{4})$$

Hence, $$a_n + b_n \cdot \sqrt[3]{2} + c_n \cdot \sqrt[3]{4} = 2^{-n/3} \sum_{k=0}^{n} {}^nC_k \cdot a_k$$

$$+ 2^{-n/3} \sum_{k=0}^{n} {}^nC_k \cdot b_k \cdot \sqrt[3]{2} + 2^{-n/3} \sum_{k=0}^{n} {}^nC_k \cdot c_k \cdot \sqrt[3]{4}$$

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1. For $r = 0, 1, \ldots\ldots, 10$, let A_r, B_r, C_r denote respectively, the coefficients of x^r in the expansion of $(1 + x)^{10}$, $(1+x)^{20}, (1+x)^{30}$. Then $\sum_{r=1}^{10} A_r (B_{10}B_r - C_{10} A_r) =$

(Nitish, West Bengal)

Ans. $(1 + x)^{10} = A_0 + A_1x + A_2x^2 + \ldots\ldots\ldots + A_{10}x^{10}$

$(x + 1)^{20} = B_0x^{20} + B_1x^{19} + B_2x^{18} + \ldots\ldots\ldots + B_{20}$.

Considering the coefficient of x^{20} in the product, we get $A_0B_0 + A_1B_1 + A_2B_2 + \ldots\ldots\ldots + A_{10}B_{10}$ = coefficient of x^{20} in the expansion of $(1 + x)^{10}(x + 1)^{20} = (1 + x)^{30}$ which is C_{20}

$$\therefore \quad \sum_{r=0}^{10} A_r B_r = C_{20} = C_{10}$$

But $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots\ldots + \binom{n}{n}^2 = \binom{2n}{n}$

$$\therefore \quad A_0^2 + A_1^2 + A_2^2 + \ldots\ldots\ldots + A_n^2 = \binom{20}{10} = B_{10}$$

$\sum_{r=0}^{10} A_r^2 = B_{10}$. Now, $\sum_{r=1}^{10} A_r (B_{10} B_r - C_{10} A_r)$

$= B_{10} \sum_{r=1}^{10} A_r B_r - C_{10} \sum_{r=1}^{10} A_r^2$

$= B_{10} (C_{10} - 1) - C_{10} (B_{10} - 1) = C_{10} - B_{10}$.

2. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$, $f(x)$ is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of $y = f(x)$ with x-axis and point B is such that the chord AB subtends a right angle at V. Find the area enclosed by $y = f(x)$ and the chord AB. *(Badal Rana, Assam)*

Ans. The given equation implies, $(4f(-1) - 3)x^2 + (4f(1) - 3)x + f(2) = 0$ is satisfied by 3 roots a, b, c

$\Rightarrow$ It is an identity $\therefore \; f(-1) = \frac{3}{4}, f(1) = \frac{3}{4}, f(2) = 0$

If $f(\alpha) = \alpha x^2 + \beta x + \gamma$, then $\beta = 0$, $\alpha = -\frac{1}{4}$, $\gamma = 1$

$\Rightarrow \quad f(x) = -\frac{x^2}{4} + 1 = \frac{4 - x^2}{4}$

The maximum point, $V = (0, 1)$; $A(-2, 0)$. Taking $B = (2t, 1 - t^2)$

$\angle AVB = \frac{\pi}{2} \Rightarrow \frac{1}{2} \cdot \left(-\frac{t}{2}\right) = -1 \Rightarrow t = 4 \therefore B = (8, -15)$

Equation of AB is $3x + 2y + 6 = 0$

The area required is $\int_{-2}^{8} \left(\frac{4 - x^2}{4} + \frac{3(x+2)}{2} \right) dx = 41.67$

3. Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from A, B, C to the major axis meet the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ respectively at P, Q, R so that P, Q, R lie on the same side of the major axis as A, B, C respectively. Prove that the normals to the ellipse at P, Q, R are concurrent. *(Manoj Chanduka, Ahmedabad)*

Ans. A, B, C are the vertices of an equilateral triangle.

$A\ (a\cos\theta, a\sin\theta)$,

$B\left(a\cos\left(\theta + \frac{2\pi}{3}\right), a\sin\left(\theta + \frac{2\pi}{3}\right)\right)$,

$C\left(a\cos\left(\theta + \frac{4\pi}{3}\right), a\sin\left(\theta + \frac{4\pi}{3}\right)\right)$

Hence, $P\ (a\cos\theta, b\sin\theta)$,

$Q\left(a\cos\left(\theta + \frac{2\pi}{3}\right), b\sin\left(\theta + \frac{2\pi}{3}\right)\right)$,

$R\left(a\cos\left(\theta + \frac{4\pi}{3}\right), b\sin\left(\theta + \frac{4\pi}{3}\right)\right)$

The normals to the ellipse at P, Q, R are respectively,

$L_1 \equiv ax\sin\theta - by\cos\theta - \frac{a^2 - b^2}{2}\sin 2\theta = 0;$

$L_2 \equiv ax\sin\left(\theta + \frac{2\pi}{3}\right) - by\cos\left(\theta + \frac{2\pi}{3}\right) - \frac{a^2 - b^2}{2}\sin^2\left(\theta + \frac{2\pi}{3}\right) = 0;$ $L_3 \equiv ax\sin\left(\theta + \frac{4\pi}{3}\right)$

$- by\cos\left(\theta + \frac{4\pi}{3}\right) - \frac{a^2 - b^2}{2}\sin 2\left(\theta + \frac{4\pi}{3}\right) = 0$

Since, $\sin\theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right) = 0$

$\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right) = 0$

$\sin 2\theta + \sin 2\left(\theta + \frac{2\pi}{3}\right) + \sin 2\left(\theta + \frac{4\pi}{3}\right) = 0$

Hence L_1, L_2, L_3 are concurrent.

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